NEW DIRECTIONS IN DUALITY THEORY FOR MODAL LOGIC

BY

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DEDICATION

I dedicate this work to my family and all the friends who helped me through these years.

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ABSTRACT

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In this work we present some new contributions towards two different directions in the study of modal logic. First we employ tense logics to provide a temporal interpretation of intuitionistic quantifiers as "always in the future" and "sometime in the past." This is achieved by modifying the Gödel translation and resolves an asymmetry between the standard interpretation of intuitionistic quantifiers.

Then we generalize the classic Gelfand-Naimark-Stone duality between compact Hausdorff spaces and uniformly complete bounded archimedean ℓ-algebras to a duality encompassing compact Hausdorff spaces with continuous relations. This leads to the notion of modal operators on bounded archimedean ℓ-algebras and in particular on rings of continuous real-valued functions on compact Hausdorff spaces. This new duality is also a generalization of the classic Jónsson-Tarski duality in modal logic.

CONTENTS

1 Introduction

This thesis presents some new contributions towards two different directions in the study of modal logic. In the first part we will employ tense logic to provide a temporal interpretation of intuitionistic quantifiers as "always in the future" and "sometime in the past." This is achieved by modifying the Gödel translation, thus resolving an asymmetry between the interpretation of intuitionistic quantifiers. This results in new tense logics that are of independent interest.

Duality theory for modal algebras yields that modal operators on boolean algebras can be modeled by continuous relations on Stone spaces. In the second part of this thesis we will show that this approach generalizes to compact Hausdorff spaces. We achieve this by generalizing Gelfand-Naimark-Stone duality between compact Hausdorff spaces and uniformly complete bounded archimedean ℓ-algebras to a duality encompassing compact Hausdorff spaces with continuous relations. This will lead us to the definition of modal operators on bounded archimedean ℓ-algebras and in particular on rings of continuous real-valued functions on a compact Hausdorff space. This new duality also generalizes the classic Jónsson-Tarski duality in modal logic.

Temporal interpretation of intuitionistic quantifiers

Intuitionism originates from the writings of Brouwer at the beginning of the twentieth century. In 1920s Heyting provided a formal framework to work with intuitionistic logic by axiomatizing it. Topological semantics was developed in 1930s by Stone [104] and Tarski [107, 108] (see also McKinsey-Tarski [92]). At the beginning of 1960s, the discovery of relational semantics revolutionized the study of intuitionistic logic. An intuitionistic frame consists of a set of worlds together with an accessibility partial order. The two quantifiers are not definable from each other in intuitionistic logic. Moreover, their interpretation in the relational semantics is asymmetric. Indeed, a world w of a model satisfies the formula $\forall x A$ iff A is true at every object of the domain D_v of every world v accessible from w, while w satisfies $\exists x A$ iff A is true at some object in the domain D_w of w. One can think of the worlds in an intuitionistic frame as states of knowledge and the accessibility order as a temporal ordering of the states. Under this interpretation, intuitionism can be thought of as the logic of the evolution of scientific knowledge. In this way we interpret the intuitionistic universal quantifier as "for every object in the future," while the existential quantifier as "for some object in the present." Thinking of the accessibility order in a temporal way can resolve the asymmetry between the two quantifiers. Indeed, it is also true that in any intuitionistic model a world w satisfies $\exists x A$ iff A is true at some object of the domain D_v of some world v from which w is accessible. Thus, the existential quantifier can be interpreted as "for some object in the past." The goal of the first part of this dissertation is to realize this temporal interpretation via translations into tense logics.

Gödel [66] defined a full and faithful translation of the intuitionistic propositional calculus IPC into the classical propositional modal system S4. This translation was studied from the point of view of the algebraic semantics by McKinsey and Tarski [93]. Heyting algebras provide an algebraic semantics for IPC and algebraic semantics for S4 is given by closure algebras, which are boolean algebras together with an operator $\Diamond$ satisfying Kuratowski axioms. The operator dual to $\Diamond$ is denoted by $\Box$. Closure algebras are also called S4-algebras in the literature on modal logic. McKinsey and Tarski showed that Heyting algebras are up to isomorphism the algebras of $\Box$-fixpoints of $\mathsf{S4}$-algebras. The motivation to study $\mathsf{S4}$-algebras comes from topology since the closure operator on a topological space satisfies the Kuratowski axioms.

There are infinitely many propositional modal logics extending $\mathsf{S4}$ into which IPC can be translated. Esakia's theorem [50] states that the logic Grz introduced by Grzegorczyk [70] is the largest one with this property. Moreover, the Blok-Esakia theorem says that the Gödel translation gives rise to a lattice isomorphism between the lattice of propositional intuitionistic logics extending IPC and the lattice of classical normal modal logics extending Grz (see, e.g, [40, p. 325]).

The Gödel translation can be extended to the predicate setting by defining

$$(\forall x A)^t = \Box \forall x A^t \qquad \text{and} \qquad (\exists x A)^t = \exists x A^t.$$

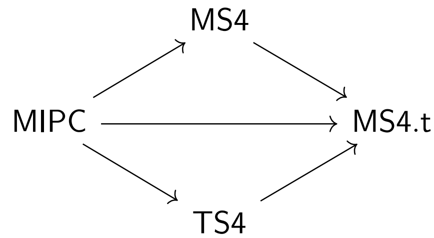
Rasiowa and Sikorski [99] showed that this extension is a full and faithful translation of the predicate intuitionistic calculus IQC into the predicate modal system $\mathsf{QS4}$. However, this translation reflects the asymmetry of the two quantifiers. We will modify the Gödel translation so that the interpretation of the existential quantifier becomes "for some object in the past." To achieve this we will employ tense logic.

Tense logic was introduced by Prior [98] to reason about events occurring at different times. Tense logics are characterized by a pair of modal operators: one for the future and one for the past. The standard relational semantics for tense logics utilizes the same frames and models as the usual relational semantics of modal logic. However, the temporal modalities are interpreted using both the accessibility relation (for the future modality) and its inverse relation (for the past modality). For more information about tense logic see [57, 67].

We first investigate a temporal translation of the monadic fragment of intuitionistic predicate logic consisting of the formulas containing only one fixed variable. Prior [98] defined the monadic intuitionistic propositional calculus MIPC and Bull [37] showed that MIPC axiomatizes the monadic fragment of IQC (see also [97]). Algebraic models of MIPC are monadic Heyting algebras introduced by Monteiro and Varsavsky [94] and studied in depth by Bezhanishvili in [10, 11, 12]. Fischer-Servi [52] studied the multimodal logic MS4 corresponding to the monadic fragment of QS4. Monadic S4-algebras are algebraic models of MS4. Fischer-Servi showed that the predicate Gödel translation restricts to a full and faithful translation of MIPC into MS4.

We introduce a tense extension of S4 which we denote by TS4. The tense modalities in TS4 are denoted by $\blacksquare_F$ and $\blacksquare_P$ and are interpreted as "always in the future" and "always in the past," respectively. The corresponding dual operators are denoted by $\blacklozenge_F$ and $\blacklozenge_P$ and are interpreted as "sometime in the future" and "sometime in the past", respectively. We define the algebraic and relational semantics for TS4 and prove completeness using canonicity. We then modify the Gödel translation by translating $\forall$ as $\blacksquare_F$ and $\exists$ as $\blacklozenge_P$. We prove that this translation embeds MIPC into TS4 fully and faithfully by utilizing the respective relational semantics. This allows us to give the desired temporal interpretation of intuitionistic monadic quantifiers as "always in the future" (for $\forall$) and "sometime in the past" (for $\exists$).

While MS4 and TS4 are not comparable, we introduce a common extension that we denote by MS4.t. The system MS4.t can be thought of as a tense extension of MS4. We provide an algebraic and relational semantics for MS4.t and prove that there exist full and faithful translations of MIPC, MS4, and TS4 into MS4.t by utilizing the respective relational semantics. Hence we obtain the following diagram, which commutes up to logical equivalence.



In addition, we prove that MS4.t has the finite model property (fmp). It is then an easy consequence of the fullness and faithfulness of the translations considered that the other systems also have the fmp.

We then move to the predicate setting where we interpret the intuitionistic universal quantifier as "for every object in the future" and the intuitionistic existential quantifier as "for some object in the past." We show that such an interpretation is supported by translating IQC fully and faithfully into a predicate tense logic by an appropriate modification of the Gödel translation. As far as we know, this approach has not been considered in the past. One obvious obstacle is that it is unclear what predicate tense logic to choose as a target for such a translation. Indeed, a natural candidate would be the standard predicate extension QS4.t of S4.t. However, since QS4.t proves the Barcan formula, and hence the Kripke frames validating QS4.t have constant domains, IQC does not translate fully into QS4.t. Instead we work with a weaker logic in which the universal instantiation axiom $\forall x A \to A(y/x)$ is weakened. This approach is along the lines of Kripke [82], Hughes and Cresswell [73], Fitting and Mendelsohn [54], and Corsi [41] who considered modal predicate logics without the Barcan and/or converse Barcan formulas. The generalized Kripke frames considered in this semantics have two domains associated to each world, an inner domain and an outer domain. The inner domains are always contained in the outer domains and are not necessarily increasing. While variables are interpreted in the outer domains, the scope of quantifiers is restricted to the inner domains. Utilizing this approach, we define a tense predicate logic $\mathsf{Q}^\circ\mathsf{S4.t}$ which is sound with respect to the generalized Kripke semantics with nonempty increasing inner domains and constant outer domains. We modify the Gödel translation to define a temporal translation of IQC into $\mathsf{Q}^\circ\mathsf{S4.t}$ by setting

$$(\forall x A)^t = \Box_F \forall x A^t \qquad \text{and} \qquad (\exists x A)^t = \Diamond_P \exists x A^t.$$

Here $\Box_F$ is the modality interpreted as "always in the future" and $\Diamond_P$ is the modality interpreted as "sometime in the past." Our main result states that this translation of IQC into $\mathsf{Q}^\circ\mathsf{S4.t}$ is full and faithful on sentences.

Modal operators on rings of continuous functions

In the second half of the twentieth century powerful mathematical tools have been developed to study modal logics. Algebraic semantics originates from the work of McKinsey and Tarski [91]. Jónsson and Tarski [77] studied boolean algebras with operators (BAOs) and began connecting the algebraic and relational semantics by obtaining the first representation results. Further results were obtained by Dummet and Lemmon [46] and Lemmon [85]. They culminated in 1970s with the birth of duality theory from the works of Esakia, Thomason, and Goldblatt. By building on the work of Stone, they showed that there is a dual equivalence between the category of modal algebras and the category of Stone spaces endowed with a continuous relation. This is known as Jónsson-Tarski duality and it allows to link algebraic and relational semantics through topology. In its present form it was established by Esakia [48] and Goldblatt [68] (but see also Halmos [71]). The Jónsson-Tarski duality can also be obtained via algebraic/coalgebraic methods. The Vietoris endofunctor on the category of Stone spaces associates to each Stone space the set of its closed subsets with a topology that makes it into a Stone space. It turns out that Stone spaces together with continuous relations can be described as coalgebras for the Vietoris functor. In [84] it is shown that one can define an endofunctor on the category of boolean algebras so that modal algebras are exactly the algebras for this endofunctor (see also [1, 63]). Since such algebras form a category that is dually equivalent to the category of coalgebras for the Vietoris functor, Jónsson-Tarski duality is obtained as a consequence.

It is often natural to drop the zero-dimensionality condition from Stone spaces and work with compact Hausdorff spaces. Dualities for the category of compact Hausdorff spaces have been studied extensively in the past, and there are different approaches that can be taken. Isbell [75] proved that the category of compact Hausdorff spaces is dually equivalent to the category of compact regular frames by associating to each space the frame of its open subsets. De Vries [44] obtained a duality between the categories of compact Hausdorff spaces and what we now call de Vries algebras by associating to each space the complete boolean algebra of its regular open subsets together with a proximity relation. In the second part of this dissertation we will be interested in dualities for compact Hausdorff spaces that are obtained by associating with each space a set of continuous functions. These are the dualities that historically appeared first. We now provide a short history of the different approaches employed to investigate rings of continuous functions, for more information see [111]. The systematic study of rings of continuous functions started in the 1930s and 1940s with the work of Stone, Gelfand, and Kolmogorov. Gelfand and Naimark [62] showed that associating to each compact Hausdorff space the ring of its continuous complex-valued functions gives rise to a dual equivalence between the category of compact Hausdorff spaces and the category of commutative C^*-algebras. Stone [106] axiomatized the rings of continuous real-valued functions on compact Hausdorff spaces. These two approaches are closely related. Indeed, the rings studied by Stone can be realized as the rings of self-adjoint elements of commutative C^*-algebras. Since each commutative C^*-algebra is isomorphic to the complexification of the ring of its self-adjoint elements, the two categories are equivalent.

Further study of rings of continuous real-valued functions was done by Kaplanski, Henriksen, Johnson, Isbell, and others. This and related topics are discussed in detail in the well-known book by Gillman and Jerison [65]. The study of continuous real-valued functions in the signature of vector lattices (without multiplication) goes back to the Krein brothers, Kakutani, Yosida, and others. Many results in this direction are collected in the well-known book by Luxemburg and Zaanen [86]. The study of these structures continues to thrive to this day.

Our interest here is in the more ring-theoretic approach. Recent contributions are due to Bezhanishvili, Morandi, and Olberding who in [24] introduced and investigated bounded archimedean ℓ-algebras that are a particular case of the structures studied by Henriksen and his collaborators in 1950s and 1960s. They showed that there is a dual adjunction between the categories of compact Hausdorff spaces KHaus and the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras. This adjunction restricts to a dual equivalence between $\mathsf{KHaus}$and the category $\boldsymbol{uba\ell}$ of uniformly complete bounded archimedean ℓ-algebras. We will refer to this duality as Gelfand-Naimark-Stone duality or simply as Gelfand duality. The well-known Stone-Weierstrass theorem and Hölder's theorem play a fundamental role in obtaining this duality. It turns out that each bounded archimedean ℓ-algebra can be embedded into a uniformly complete one. Moreover, each uniformly complete bounded archimedean ℓ-algebra is isomorphic to the algebra of continuous real-valued functions over some compact Hausdorff space. The research on bounded archimedean ℓ-algebras turned out to be fruitful (see, e.g., [22, 25, 28, 29, 30]).

Isbell and de Vries dualities have been generalized to encompass continuous relations on compact Hausdorff spaces in [15, 16]. For some time now there has been a desire to obtain an analogous generalization of Gelfand-Naimark-Stone duality but it remained elusive for at least two reasons. On the conceptual side, there was no agreement on what should be the definition of modal operators on the ring $C(X)$ of continuous real-valued functions on a compact Hausdorff space X. On the technical side, it was unclear how to axiomatize attempted definitions of modal operators. Both of these obstacles will be overcome by our approach.

We call a compact Hausdorff space X together with a continuous relation R a compact Hausdorff frame. We denote the category of compact Hausdorff frames by KHF. If $(X, R) \in \mathsf{KHF}$ and $f \in C(X)$, we define the map $\Box_R f$ on X by setting

$$(\Box_R f)(x) = \begin{cases} \inf fR[x] & \text{if } R[x] \neq \varnothing \\ 1 & \text{otherwise} \end{cases}$$

for each $x \in X$, where $R[x] = \{y \in X \mid xRy\}$. We axiomatize the operator $\Box_R$ on $C(X)$ to define modal operators on bounded archimedean ℓ-algebras. We denote by $\boldsymbol{mbal}$ the resulting category of bounded archimedean ℓ-algebras equipped with a modal operator. We show that the dual adjunction described in [24] extends to a dual adjunction between the categories KHF and $\boldsymbol{mbal}$. This dual adjunction restrict to a dual equivalence between the categories KHF and the full subcategory $\boldsymbol{mubal}$ of uniformly complete algebras in $\boldsymbol{mbal}$.

Following an approach similar to the one in [84], we show that the dual adjunction between $\boldsymbol{mba\ell}$ and KHF can be obtained via algebraic/coalgebraic methods. The Vietoris space can be defined for any compact Hausdorff space. Thus, the Vietoris endofunctor is well defined on KHaus. It is well known that KHF is isomorphic to the category of coalgebras for the Vietoris functor over KHaus. We define an endofunctor $\mathcal{H}$ on $\boldsymbol{ba\ell}$ such that $\boldsymbol{mba\ell}$ is isomorphic to the category of algebras for $\mathcal{H}$. In order to define $\mathcal{H}$ we need to investigate free objects in $\boldsymbol{ba\ell}$. Although free objects over sets do not exist in $\boldsymbol{ba\ell}$, free objects over weighted sets do exist. We then show that the dual adjunction between $\boldsymbol{ba\ell}$ and KHaus extends to a dual adjunction between the categories of algebras for $\mathcal{H}$ and the category of coalgebras for the Vietoris $\mathcal{V}$ functor on KHaus. This yields an alternate way of obtaining the dual adjunction between $\boldsymbol{mba\ell}$ and KHF. Moreover, we define an endofunctor $\mathcal{H}^u$ on $\boldsymbol{uba\ell}$ such that $\boldsymbol{muba\ell}$ is isomorphic to the category of algebras for $\mathcal{H}^u$. The dual equivalence between $\boldsymbol{uba\ell}$ and KHaus extends to a dual equivalence between the categories of algebras for $\mathcal{H}^u$ and the category of coalgebras for $\mathcal{V}$ yielding an alternate way of obtaining the dual equivalence between $\boldsymbol{muba\ell}$ and KHF.

Content

Sections 2 and 3 are based on [18]. In Section 2 we define the monadic intuitionistic logic MIPC and the monadic $\mathsf{S4}$ logic $\mathsf{MS4}$. We provide their algebraic and relational semantics and give an alternate proof that the Gödel translation from MIPC into $\mathsf{MS4}$ is full and faithful. In Section 3 we define $\mathsf{TS4}$ and we prove that the temporal translation of MIPC into $\mathsf{TS4}$ is full and faithful. We then define the logic $\mathsf{MS4.t}$ and we obtain a diagram of translations that is commutative up to logical equivalence.

The content of Section 4 is based on [17]. We provide the necessary background about predicate intuitionistic and modal logics and the predicate version of the Gödel translation. We then define a predicate temporal translation of IQC into the new temporal predicate system $\mathsf{Q^{\circ}S4.t}$ and we show it is full and faithful.

Sections 5, which is based on [20], provides the necessary background about bounded archimedean ℓ-algebras and Gelfand-Naimark-Stone duality and contains our new results about modal operators on bounded archimedean ℓ-algebras and the resulting duality that generalized both Jónsson-Tarski duality and Gelfand-Naimark-Stone duality. Section 6 talks about the algebraic/coalgebraic approach to Gelfand-Naimark-Stone duality and is based on [21, 22]. We explain in detail how to overcome an obstacle in the construction of the desired endofunctor on $\boldsymbol{ba\ell}$ due to the nonexistence of free objects in $\boldsymbol{ba\ell}$ over sets.

Part I

Temporal interpretation of intuitionistic quantifiers

2 Monadic Gödel translation

In this section we review some well-known facts about the Gödel translation of MIPC into the monadic fragment of the predicate S4 logic that we denote by MS4. After providing the axiomatizations of MIPC and MS4, we define their algebraic and relational semantics. We use canonicity of the two logical systems to prove their completeness with respect to the relational semantics. We end the section by providing an alternate proof that the Gödel translation restricts to a full and faithful translation of MIPC into MS4 using the relational semantics.

2.1 MIPC

Let $\mathcal{L}$ be a propositional language and let $\mathcal{L}_{\forall\exists}$ be the extension of $\mathcal{L}$ with two modalities $\forall$ and $\exists$.

Definition 2.1. The *monadic intuitionistic propositional calculus* MIPC is the intuitionistic modal logic in the propositional modal language $\mathcal{L}_{\forall\exists}$ containing

1. all theorems of the intuitionistic propositional calculus IPC (see, e.g, [60, p. 6]);
2. the S4-axioms for $\forall$:
   (a) $\forall(p \wedge q) \leftrightarrow (\forall p \wedge \forall q)$,
   (b) $\forall p \to p$,
   (c) $\forall p \to \forall\forall p$;
3. the S5-axioms for $\exists$:

(a) $\exists(p \vee q) \leftrightarrow (\exists p \vee \exists q)$,

(b) $p \to \exists p$,

(c) $\exists\exists p \to \exists p$,

(d) $(\exists p \wedge \exists q) \to \exists(\exists p \wedge q)$;

4. the axioms connecting $\forall$ and $\exists$:

(a) $\exists\forall p \leftrightarrow \forall p$,

(b) $\exists p \leftrightarrow \forall\exists p$;

and closed under the rules of modus ponens, substitution, and necessitation ($\varphi/\forall\varphi$).

Remark 2.2.

1. There are a number of axioms that are equivalent to axiom (3d) (see, e.g., [10, Lem. 2(d)]).

2. The two modalities $\forall$ and $\exists$ are not definable from each other. Furthermore, there is an asymmetry in the axioms that the two satisfy. Indeed, the formula $\forall(\forall p \vee q) \to (\forall p \vee \forall q)$, that is the $\forall$-analogue of axiom (3d), is not a theorem of MIPC.

2.1.1 Monadic Heyting algebras

The algebraic semantics for MIPC is given by monadic Heyting algebras. These algebras were first introduced by Monteiro and Varsavsky [94] as a generalization of monadic (boolean) algebras of Halmos [71]. For a detailed study of monadic Heyting algebras we refer to [10, 11, 12].

Definition 2.3. Let H be a Heyting algebra.

1. A unary function $\mathsf{i} : H \to H$ is an *interior operator* on H if

   (a) $\mathsf{i}(a \wedge b) = \mathsf{i}a \wedge \mathsf{i}b$,

   (b) $\mathsf{i}1 = 1$,

   (c) $\mathsf{i}a \leq a$,

   (d) $\mathsf{i}a \leq \mathsf{i}\mathsf{i}a$.

2. A unary function $\mathsf{c} : H \to H$ is a *closure operator* on H if

   (a) $\mathsf{c}(a \vee b) = \mathsf{c}a \vee \mathsf{c}b$,

   (b) $\mathsf{c}0 = 0$,

   (c) $a \leq \mathsf{c}a$,

   (d) $\mathsf{c}\mathsf{c}a \leq \mathsf{c}a$.

Definition 2.4. A *monadic Heyting algebra* is a triple $\mathfrak{A} = (H, \forall, \exists)$ where H is a Heyting algebra, $\forall$ is an interior operator on H, and $\exists$ is a closure operator on H satisfying:

1. $\exists(\exists a \wedge b) = \exists a \wedge \exists b$,

2. $\forall\exists a = \exists a$,

3. $\exists\forall a = \forall a$.

Let MHA be the variety of all monadic Heyting algebras.

Remark 2.5. Let $(H, \forall, \exists)$ be a monadic Heyting algebra.

1. Definition 2.4(1) has a number of equivalent conditions (see, e.g., [10, Lem. 2(d)]). These together with the conditions connecting $\forall$ and $\exists$ yield that the fixpoints of $\forall$

form a subalgebra H_0 of H which coincides with the subalgebra of the fixpoints of $\exists$. Moreover, $\forall$ and $\exists$ are the right and left adjoints of the identity embedding $H_0 \to H$, and up to isomorphism each monadic Heyting algebra arises this way (see, e.g., [10, Sec. 3]).

2. The non-symmetry of $\forall$ and $\exists$ is manifested by the fact that the $\forall$-analogue $\forall(\forall a \vee b) = \forall a \vee \forall b$ of Definition 2.4(1) does not hold in general.

The standard Lindenbaum-Tarski construction (see, e.g., [100, Ch. VI]) yields that monadic Heyting algebras provide a sound and complete algebraic semantics for MIPC.

2.1.2 Relational semantics

We now turn to the relational semantics for MIPC. There are several such (see, e.g., [11]), but we concentrate on the one introduced by Ono [95].

Definition 2.6. An MIPC-*frame* is a triple $\mathfrak{F} = (X, R, Q)$ where X is a set, R is a partial order, Q is a quasi-order (reflexive and transitive), and the following two conditions are satisfied:

(O1) $R \subseteq Q$,

(O2) $xQy \Rightarrow (\exists z)(xRz \;\&\; zE_Qy)$.

Here E_Q is the equivalence relation defined by xE_Qy iff xQy and yQx.

Let $\mathfrak{F} = (X, R, Q)$ be an MIPC-frame. As usual, for $x \in X$, we write

$$R[x] = \{y \in X \mid xRy\} \text{ and } R^{-1}[x] = \{y \in X \mid yRx\},$$

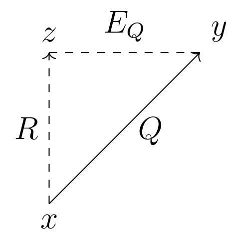


Figure 1: Condition (O2).

and for $U \subseteq X$, we write

$$R[U] = \bigcup\{R[u] \mid u \in U\} \text{ and } R^{-1}[U] = \bigcup\{R^{-1}[u] \mid u \in U\}.$$

We use the same notation for Q and E_Q. Since E_Q is an equivalence relation, we have that $E_Q[x] = (E_Q)^{-1}[x]$ and $E_Q[U] = (E_Q)^{-1}[U]$.

We call a subset U of X an R-*upset* provided $U = R[U]$ ($x \in U$ and xRy imply $y \in U$). Let $\mathsf{Up}(X)$ be the set of all R-upsets of $\mathfrak{F}$. It is well known that $\mathsf{Up}(X)$ is a Heyting algebra, where the lattice operations are set-theoretic union and intersection, and $U \to V$ is calculated by

$$U \to V = \{x \in X \mid R[x] \cap U \subseteq V\} = X \setminus R^{-1}[U \setminus V].$$

In addition, for $U \in \mathsf{Up}(X)$, define

$$\forall_Q(U) = X \setminus Q^{-1}[X \setminus U] \text{ and } \exists_Q(U) = E_Q[U].$$

Then $\mathfrak{F}^+ = (\mathsf{Up}(X), \forall_Q, \exists_Q)$ is a monadic Heyting algebra (see, e.g., [11, Sec. 6]).

Remark 2.7. If $U \in \mathsf{Up}(X)$, then Definition 2.6(O2) implies that $E_Q[U] = Q[U]$. That $\exists_Q(U) = Q[U]$ motivates our interpretation of $\exists$ as "sometime in the past." Indeed, taking $Q[U]$ is the standard way to associate an operator on $\wp(X)$ to the tense modality "sometime in the past" (see, e.g., [110, p. 151]). As a consequence of this, $(\mathfrak{F}^+)_0$ is the set of Q-upsets of $\mathfrak{F}$.

Each monadic Heyting algebra $\mathfrak{A} = (H, \forall, \exists)$ can be represented as a subalgebra of $\mathfrak{F}^+$ for some MIPC-frame $\mathfrak{F}$. For this we recall the definition of the canonical frame of $\mathfrak{A}$.

Definition 2.8. Let $\mathfrak{A} = (H, \forall, \exists)$ be a monadic Heyting algebra. The *canonical frame* of $\mathfrak{A}$ is the frame $\mathfrak{A}_+ = (X_\mathfrak{A}, R_\mathfrak{A}, Q_\mathfrak{A})$ where $X_\mathfrak{A}$ is the set of prime filters of H, $R_\mathfrak{A}$ is the inclusion relation, and $xQ_\mathfrak{A}y$ iff $x \cap H_0 \subseteq y$ (equivalently, $x \cap H_0 \subseteq y \cap H_0$).

By [11, Sec. 6], $\mathfrak{A}_+$ is an MIPC-frame.

Definition 2.9. We call an MIPC-frame $\mathfrak{F}$ *canonical* if it is isomorphic to $\mathfrak{A}_+$ for some monadic Heyting algebra $\mathfrak{A}$.

Define the *Stone map* $\beta : \mathfrak{A} \to \mathsf{Up}(X_\mathfrak{A})$ by

$$\beta(a) = \{x \in X_\mathfrak{A} \mid a \in x\}.$$

By [11, Sec. 6], $\beta : \mathfrak{A} \to (\mathfrak{A}_+)^+$ is a one-to-one homomorphism of monadic Heyting algebras. Thus, we arrive at the following representation theorem for monadic Heyting algebras.

Proposition 2.10. *Each monadic Heyting algebra $\mathfrak{A}$ is isomorphic to a subalgebra of $(\mathfrak{A}_+)^+$.*

Remark 2.11.

1. The image of $\mathfrak{A}$ inside $(\mathfrak{A}_+)^+$ can be recovered by introducing a Priestley topology on $X_\mathfrak{A}$. This leads to the notion of *perfect* MIPC*-frames* and a duality between the category of monadic Heyting algebras and the category of perfect MIPC-frames; see [11, Thm. 17].

2. When $\mathfrak{A}$ is finite, its embedding into $(\mathfrak{A}_+)^+$ is an isomorphism, and hence the categories of finite monadic Heyting algebras and finite MIPC-frames are dually equivalent.

The next corollary is an immediate consequence of the above considerations.

Corollary 2.12. MIPC *is canonical; that is,*

$$\mathfrak{A} \in \mathsf{MHA} \Rightarrow (\mathfrak{A}_+)^+ \in \mathsf{MHA}.$$

A *valuation* on an MIPC-frame $\mathfrak{F} = (X, R, Q)$ is a map v associating an R-upset of X to any propositional letter of $\mathcal{L}_{\forall\exists}$. The connectives $\wedge, \vee, \to, \neg$ are then interpreted as in intuitionistic Kripke frames, and $\forall, \exists$ are interpreted by

$$\begin{array}{lll} x \vDash_v \forall\varphi & \text{iff} & (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v \exists\varphi & \text{iff} & (\exists y \in X)(xE_Q y \ \&\ y \vDash_v \varphi). \end{array}$$

We say that φ is *valid* in $\mathfrak{F}$, and write $\mathfrak{F} \vDash \varphi$, if $x \vDash_v \varphi$ for every valuation v and every $x \in X$.

Theorem 2.13. $\mathsf{MIPC} \vdash \varphi$ *iff* $\mathfrak{F} \vDash \varphi$ *for every* MIPC*-frame* $\mathfrak{F}$.

Proof. Soundness of MIPC with respect to this semantics is straightforward to prove. For completeness, suppose that $\mathsf{MIPC} \nvdash \varphi$. By algebraic completeness, there is a monadic Heyting algebra $\mathfrak{A}$ such that $\mathfrak{A} \nvDash \varphi$. Since $\mathfrak{A}$ is isomorphic to a subalgebra of $(\mathfrak{A}_+)^+$, we have $(\mathfrak{A}_+)^+ \nvDash \varphi$. Thus, $\mathfrak{A}_+$ is an MIPC-frame such that $\mathfrak{A}_+ \nvDash \varphi$. □

We conclude this section by recalling that MIPC has the fmp. This was first established by Bull [36] using algebraic semantics. His proof contained a gap, which was corrected independently by Fischer-Servi [53] and Ono [95]. A semantic proof is given in [58], which is based on the technique developed by Grefe [69]. We will give yet another proof of this result in Section 3.5.

2.2 MS4

We now recall the definition of the monadic $\mathsf{S4}$ logic $\mathsf{MS4}$. Let $\mathcal{L}_{\Box\forall}$ be a propositional bimodal language with two modal operators $\Box$ and $\forall$.

Definition 2.14. The *monadic* $\mathsf{S4}$, denoted $\mathsf{MS4}$, is the smallest classical bimodal logic containing the $\mathsf{S4}$-axioms for $\Box$, the $\mathsf{S5}$-axioms for $\forall$, the left commutativity axiom

$$\Box\forall p \to \forall\Box p,$$

and closed under modus ponens, substitution, $\Box$-necessitation, and $\forall$-necessitation.

As usual, $\Diamond$ is an abbreviation for $\neg\Box\neg$ and $\exists$ is an abbreviation for $\neg\forall\neg$.

Remark 2.15. Recalling the definition of fusion of two logics (see [58]), $\mathsf{MS4}$ is obtained from the fusion $\mathsf{S4} \otimes \mathsf{S5}$ by adding the left commutativity axiom $\Box\forall p \to \forall\Box p$ which is the monadic version of the converse Barcan formula. The monadic version of the Barcan formula is the right commutativity axiom $\forall\Box p \to \Box\forall p$. Adding it to $\mathsf{MS4}$ yields the product logic $\mathsf{S4} \times \mathsf{S5}$; see [58, Ch. 5] for details.

2.2.1 Monadic S4 algebras

The algebraic semantics for $\mathsf{MS4}$ is given by monadic $\mathsf{S4}$-algebras. To define these algebras, we first recall the definition of $\mathsf{S4}$-algebras and $\mathsf{S5}$-algebras.

Definition 2.16.

1. An $\mathsf{S4}$-*algebra*, or an *interior algebra*, is a pair $\mathfrak{B} = (B, \Box)$ where B is a boolean algebra and $\Box$ is an interior operator on B (see Definition 2.3(1)).

2. An $\mathsf{S5}$-*algebra*, or a *monadic algebra*, is an $\mathsf{S4}$-algebra $\mathfrak{B} = (B, \forall)$ that in addition satisfies $a \leq \forall\exists a$ for all $a \in B$.

Remark 2.17. $\mathsf{S4}$-algebras were first introduced by McKinsey and Tarski [91]. They worked with the closure operator $\Diamond$ dual to $\Box$ and hence they called them *closure algebras*. Rasiowa and Sikorski [100] switched to $\Box$ and called them *topological boolean algebras*. Blok [35] called them *interior algebras*. $\mathsf{S5}$-algebras were defined by Halmos [71] who called them *monadic algebras*. The names $\mathsf{S4}$-algebra and $\mathsf{S5}$-algebra became standard in the modal logic literature of the end of the twentieth century and the beginning of the twenty-first century.

We are ready to define monadic $\mathsf{S4}$-algebras.

Definition 2.18. A *monadic* $\mathsf{S4}$-*algebra*, or an $\mathsf{MS4}$-*algebra* for short, is a tuple $\mathfrak{B} = (B, \Box, \forall)$ where

1. $(B, \Box)$ is an $\mathsf{S4}$-algebra,
2. $(B, \forall)$ is an $\mathsf{S5}$-algebra,
3. $\Box\forall a \leq \forall\Box a$ for each $a \in B$.

Lemma 2.19. *The axiom $\Box\forall a \leq \forall\Box a$ in Definition 2.18 can be replaced by any of the following:*

1. $\Box\forall\Box a = \Box\forall a$.
2. $\forall\Box\forall a = \Box\forall a$.
3. $\exists\Box\exists a = \Box\exists a$.

4. $\Box\exists\Box a = \exists\Box a$.

5. $\exists\Box a \leq \Box\exists a$.

Proof. Showing that (1) and (2) are equivalent to $\Box\forall a \leq \forall\Box a$ is straightforward. That (3) and (4) are equivalent to (5) can be proved similarly. We show that (2) and (3) are equivalent. Suppose (2) holds. Then for each $a \in B$, we have

$$\forall\Box\exists a = \forall\Box\forall\exists a = \Box\forall\exists a = \Box\exists a.$$

Using $\forall\Box\exists a = \Box\exists a$ twice, we obtain

$$\exists\Box\exists a = \exists\forall\Box\exists a = \forall\Box\exists a = \Box\exists a,$$

yielding (3). Proving (2) from (3) is analogous. □

Remark 2.20. As noted above, the inequality $\Box\forall a \leq \forall\Box a$ is equivalent to the equality $\forall\Box\forall a = \Box\forall a$. This yields that the set B_0 of $\forall$-fixpoints of an $\mathsf{MS4}$-algebra $(B, \Box, \forall)$ forms an $\mathsf{S4}$-subalgebra of $(B, \Box)$ such that $\forall$ is the right adjoint to the identity embedding $B_0 \to B$. Moreover, up to isomorphism each $\mathsf{MS4}$-algebra arises this way. This is similar to the case of monadic Heyting algebras (see Remark 2.5(1)).

The Lindenbaum-Tarski construction yields that $\mathsf{MS4}$-algebras provide a sound and complete algebraic semantics for $\mathsf{MS4}$.

2.2.2 Relational semantics

The relational semantics for $\mathsf{MS4}$ was first introduced by Esakia [51].

Definition 2.21. An $\mathsf{MS4}$-*frame* is a triple $\mathfrak{F} = (X, R, E)$ where X is a set, R is a quasi-order, E is an equivalence relation, and the following commutativity condition is satisfied:

$$(\forall x, y, z \in X)(xEy \ \& \ yRz) \Rightarrow (\exists u \in X)(xRu \ \& \ uEz). \tag{E}$$

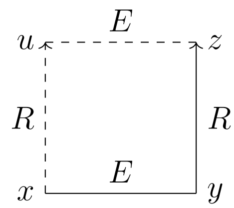


Figure 2: Condition (E).

A *valuation* on an $\mathsf{MS4}$-frame $\mathfrak{F} = (X, R, E)$ is a map v associating a subset of X to each propositional letter of $\mathcal{L}_{\Box\forall}$. Then the boolean connectives are interpreted as usual,

$$\begin{array}{lll} x \vDash_v \Box\varphi & \text{iff} & (\forall y \in X)(xRy \ \Rightarrow \ y \vDash_v \varphi), \\ x \vDash_v \forall\varphi & \text{iff} & (\forall y \in X)(xEy \ \Rightarrow \ y \vDash_v \varphi). \end{array}$$

We say that φ is *valid* in $\mathfrak{F}$, in symbols $\mathfrak{F} \vDash \varphi$, if $x \vDash_v \varphi$ for every valuation v and $x \in X$.

As a consequence of Lemma 2.19, the axiom $\Box\forall p \to \forall\Box p$ can be replaced by the axiom $\exists\Box p \to \Box\exists p$. Thus, $\mathsf{MS4}$ can be axiomatized by Sahlqvist formulas (see, e.g, [34, Sec. 3.6]). This yields the following theorem (see, e.g., [34, Thm. 4.42]):

Theorem 2.22. $\mathsf{MS4}$ *is canonical and hence is complete with respect to the relational semantics, i.e.*

$$\mathsf{MS4} \vdash \varphi \quad \textit{iff} \quad \mathfrak{F} \vDash \varphi \ \textit{for every} \ \mathsf{MS4}\textit{-frame} \ \mathfrak{F}.$$

In addition, $\mathsf{MS4}$ has the fmp and is decidable. This can be derived from the results in [59, Sec. 12] (see also [58, Thms. 6.52, 9.12]). As we will see in Section 3.5, this result also follows from the fmp of a stronger multimodal system.

We conclude this section by proving a representation theorem for $\mathsf{MS4}$-algebras. For an $\mathsf{MS4}$-frame $\mathfrak{F} = (X, R, E)$, let $\wp(X)$ be the powerset of X and for $U \in \wp(X)$ let

$$\Box_R(U) = X \setminus R^{-1}[X \setminus U] \text{ and } \forall_E(U) = X \setminus E[X \setminus U].$$

Since R is a quasi-order, $(\wp(X), \Box_R)$ is an $\mathsf{S4}$-algebra; and since E is an equivalence relation, $(\wp(X), \forall_E)$ is an $\mathsf{S5}$-algebra (see [77, Thm. 3.5]). In addition, the commutativity condition yields that $\mathfrak{F}^+ := (\wp(X), \Box_R, \forall_E)$ is an $\mathsf{MS4}$-algebra.

In fact, as in the case of monadic Heyting algebras, each $\mathsf{MS4}$-algebra $\mathfrak{B} = (B, \Box, \forall)$ is isomorphic to a subalgebra of $\mathfrak{F}^+$ for some $\mathsf{MS4}$-frame $\mathfrak{F}$. We can take $\mathfrak{F}$ to be the canonical frame of $\mathfrak{B}$. Let H be the set of $\Box$-fixpoints and B_0 the set of $\forall$-fixpoints. Then H is a Heyting algebra which is a bounded sublattice of B, and B_0 is an $\mathsf{S4}$-subalgebra of $(B, \Box)$.

Remark 2.23. If $\mathfrak{B} = \mathfrak{F}^+$, then the elements of H are the R-upsets of $\mathfrak{F}$ and the elements of B_0 are the E-saturated subsets of $\mathfrak{F}$ (that is, unions of E-equivalence classes).

Definition 2.24. Let $\mathfrak{B} = (B, \Box, \forall)$ be an $\mathsf{MS4}$-algebra. The *canonical frame* of $\mathfrak{B}$ is the frame $\mathfrak{B}_+ = (X_\mathfrak{B}, R_\mathfrak{B}, E_\mathfrak{B})$ where $X_\mathfrak{B}$ is the set of ultrafilters of B, $xR_\mathfrak{B}y$ iff $x \cap H \subseteq y$ (equivalently, $x \cap H \subseteq y \cap H$), and $xE_\mathfrak{B}y$ iff $x \cap B_0 = y \cap B_0$.

Lemma 2.25. *If $\mathfrak{B}$ is an $\mathsf{MS4}$-algebra, then $\mathfrak{B}_+$ is an $\mathsf{MS4}$-frame.*

Proof. Since $(B, \Box)$ is an $\mathsf{S4}$-algebra, $R_\mathfrak{B}$ is a quasi-order (see [77, Thm. 3.14]); and since $(B, \forall)$ is an $\mathsf{S5}$-algebra, $E_\mathfrak{B}$ is an equivalence relation (see [77, Thm. 3.18]). It remains to show that Definition 2.21(E) is satisfied. Let $x, y, z \in X_\mathfrak{B}$ be such that $xE_\mathfrak{B}y$ and $yR_\mathfrak{B}z$. This means that $x \cap B_0 = y \cap B_0$ and $y \cap H \subseteq z$. Let F be the filter of $\mathfrak{B}$ generated by $(x \cap H) \cup (z \cap B_0)$. We show that F is proper. Otherwise, since $x \cap H$ and $z \cap B_0$ are closed under meets, there are $a \in x \cap H$ and $b \in z \cap B_0$ such that $a \wedge b = 0$. Therefore, $a \leq \neg b$. Thus, $a = \Box a \leq \Box\neg b$, so $\Box\neg b \in x$. Since B_0 is an $\mathsf{S4}$-subalgebra of $(B, \Box)$ and $b \in B_0$, we have $\Box\neg b \in B_0$. This yields $\Box\neg b \in x \cap B_0 = y \cap B_0$, which implies $\Box\neg b \in y \cap H \subseteq z$. Therefore, $\neg b \in z$ which contradicts $b \in z$. Thus, F is proper, and so there is an ultrafilter u of B such that $F \subseteq u$. Consequently, $x \cap H \subseteq u$ and $z \cap B_0 \subseteq u \cap B_0$. Since $z \cap B_0$ and $u \cap B_0$ are both ultrafilters of B_0, we conclude that $z \cap B_0 = u \cap B_0$. Thus, there is $u \in X_{\mathfrak{B}}$ with $xR_{\mathfrak{B}}u$ and $uE_{\mathfrak{B}}z$. □

Definition 2.26. We call an $\mathsf{MS4}$-frame *canonical* if it is isomorphic to $\mathfrak{B}_+$ for some $\mathsf{MS4}$-algebra $\mathfrak{B}$.

For an $\mathsf{MS4}$-algebra $\mathfrak{B}$, it follows from [77, Thm. 3.14] that the Stone map $\beta : B \to \wp(X_{\mathfrak{B}})$ is a one-to-one homomorphism of $\mathsf{MS4}$-algebras. Thus, we arrive at the following representation theorem.

Proposition 2.27. *Each $\mathsf{MS4}$-algebra $\mathfrak{B}$ is isomorphic to a subalgebra of $(\mathfrak{B}_+)^+$.*

Remark 2.28.

1. To recover the image of $\mathfrak{B}$ in $\wp(X_{\mathfrak{B}})$ we need to endow $X_{\mathfrak{B}}$ with a Stone topology. This leads to the notion of *perfect* $\mathsf{MS4}$-*frames* and a duality between the category of $\mathsf{MS4}$-algebras and the category of perfect $\mathsf{MS4}$-frames which generalizes Esakia duality for $\mathsf{S4}$-algebras. This situation is analogous to the one for monadic Heyting algebras and perfect MIPC-frames (see Remark 2.11).

2. When $\mathfrak{B}$ is finite, its embedding into $(\mathfrak{B}_+)^+$ is an isomorphism, and hence the categories of finite $\mathsf{MS4}$-algebras and finite $\mathsf{MS4}$-frames are dually equivalent.

2.3 Gödel translation of MIPC into MS4

We recall that the Gödel translation of MIPC into MS4 is defined by

$$
\begin{array}{rcll}
\bot^t & = & \bot & \\
p^t & = & \Box p & \text{for each propositional letter } p \\
(\varphi \wedge \psi)^t & = & \varphi^t \wedge \psi^t & \\
(\varphi \vee \psi)^t & = & \varphi^t \vee \psi^t & \\
(\varphi \to \psi)^t & = & \Box(\neg\varphi^t \vee \psi^t) & \\
(\forall\varphi)^t & = & \Box\forall\varphi^t & \\
(\exists\varphi)^t & = & \exists\varphi^t &
\end{array}
$$

It was shown by Fischer-Servi [52] that this translation is full and faithful, meaning that

$$\mathsf{MIPC} \vdash \varphi \text{ iff } \mathsf{MS4} \vdash \varphi^t.$$

Fischer-Servi used the translations of MIPC and MS4 into IQC and QS4 respectively, and the predicate version of the Gödel translation. In [53] she gave a different proof of this result using the fmp for MIPC. We give yet another proof utilizing relational semantics for MIPC and MS4. Our proof generalizes the semantic proof that the Gödel translation of IPC into S4 is full and faithful (see, e.g., [40, Sec. 3.9]). We require the following lemma.

Lemma 2.29. *For any formula χ of $\mathcal{L}_{\forall\exists}$, we have*

$$\mathsf{MS4} \vdash \chi^t \to \Box\chi^t.$$

Proof. We first show that $\mathsf{MS4} \vdash \exists\Box\varphi \to \Box\exists\varphi$ for any formula φ of $\mathcal{L}_{\Box\forall}$. For this, by algebraic completeness, it is sufficient to prove that the inequality $\exists\Box a \leq \Box\exists a$ holds in every MS4-algebra $(B, \Box, \forall)$. Let $a \in B$. We have

$$\exists\Box a \leq \exists\Box\exists a = \exists\Box\forall\exists a \leq \exists\forall\Box\exists a = \forall\Box\exists a \leq \Box\exists a.$$

We are now ready to prove that $\mathsf{MS4} \vdash \chi^t \to \Box\chi^t$ by induction on the complexity of χ. This is obvious when $\chi = \bot$. The cases when χ is p, $\varphi \to \psi$, or $\forall\varphi$ follow from the axiom $\Box\varphi \to \Box\Box\varphi$. We next consider the cases when χ is $\varphi \wedge \psi$ or $\varphi \vee \psi$. Suppose that the claim is true for φ and ψ, so $\varphi^t \to \Box\varphi^t$ and $\psi^t \to \Box\psi^t$ are theorems of $\mathsf{MS4}$. Then $\varphi^t \wedge \psi^t \to \Box(\varphi^t \wedge \psi^t)$ and $\varphi^t \vee \psi^t \to \Box(\varphi^t \vee \psi^t)$ are also theorems of $\mathsf{MS4}$. Finally, if χ is $\exists\varphi$ and $\mathsf{MS4} \vdash \varphi^t \to \Box\varphi^t$, then $\mathsf{MS4} \vdash \exists\varphi^t \to \exists\Box\varphi^t$. Therefore, since $\mathsf{MS4} \vdash \exists\Box\varphi^t \to \Box\exists\varphi^t$, we conclude that $\mathsf{MS4} \vdash \exists\varphi^t \to \Box\exists\varphi^t$. $\Box$

In the next definition we generalize to $\mathsf{MS4}$-frames the well-known definition of skeleton (see, e.g., [40, Sec. 3.9]).

Definition 2.30. Let $\mathfrak{F} = (X, R, E)$ be an $\mathsf{MS4}$-frame. Define the relation Q_E on X by setting xQ_Ey iff $(\exists z \in X)(xRz \ \& \ zEy)$. Then the *skeleton* $\mathfrak{F}^t = (X', R', Q')$ of $\mathfrak{F}$ is defined as follows. Let $\sim$ be the equivalence relation on X given by $x \sim y$ iff xRy and yRx. We let X' be the set of equivalence classes of $\sim$, and define R' and Q' on X' by $[x]R'[y]$ iff xRy and $[x]Q'[y]$ iff xQ_Ey.

Proposition 2.31.

1. *If $\mathfrak{F}$ is an $\mathsf{MS4}$-frame, then $\mathfrak{F}^t$ is an MIPC-frame.*

2. *For each valuation v on $\mathfrak{F}$ there is a valuation v' on $\mathfrak{F}^t$ such that for each $x \in \mathfrak{F}$ and $\mathcal{L}_{\forall\exists}$-formula φ, we have*

$$\mathfrak{F}^t, [x] \vDash_{v'} \varphi \text{ iff } \mathfrak{F}, x \vDash_v \varphi^t.$$

3. *For each $\mathcal{L}_{\forall\exists}$-formula φ, we have*

$$\mathfrak{F}^t \vDash \varphi \text{ iff } \mathfrak{F} \vDash \varphi^t.$$

4. *For each MIPC-frame $\mathfrak{G}$ there is an $\mathsf{MS4}$-frame $\mathfrak{F}$ such that $\mathfrak{G}$ is isomorphic to $\mathfrak{F}^t$.*

Proof. (1). It is well known that (X', R') is an intuitionistic Kripke frame. That Q' is well defined follows from Condition (E). Showing that Q' is a quasi-order, and that (O1) and (O2) hold in $\mathfrak{F}^t$ is straightforward.

(2). Define v' on $\mathfrak{F}^t$ by $v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\}$. We show that $\mathfrak{F}^t, [x] \vDash_{v'} \varphi$ iff $\mathfrak{F}, x \vDash_v \varphi^t$ by induction on the complexity of φ. Since $v'(p) = \{[x] \mid \mathfrak{F}, x \vDash_v \Box p\}$, the claim is obvious when φ is a propositional letter. We prove the claim for φ of the form $\forall\psi$ and $\exists\psi$ since the other cases are well known. Suppose $\varphi = \forall\psi$. By the definition of Q' and induction hypothesis, we have

$$\begin{aligned}\mathfrak{F}^t, [x] \vDash_{v'} \forall\psi &\text{ iff } (\forall [y] \in X')([x]Q'[y] \Rightarrow \mathfrak{F}^t, [y] \vDash_{v'} \psi)\\ &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}^t, [y] \vDash_{v'} \psi)\\ &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}, y \vDash_v \psi^t).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathfrak{F}, x \vDash_v (\forall\psi)^t &\text{ iff } \mathfrak{F}, x \vDash_v \Box\forall\psi^t\\ &\text{ iff } (\forall z \in X)(xRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}, y \vDash_v \psi^t))\\ &\text{ iff } (\forall y \in X)(xQ_E y \Rightarrow \mathfrak{F}, y \vDash_v \psi^t).\end{aligned}$$

Thus, $\mathfrak{F}^t, [x] \vDash_{v'} \forall\psi$ iff $\mathfrak{F}, x \vDash_v (\forall\psi)^t$.

Suppose $\varphi = \exists\psi$. As noted in Remark 2.7, Q' and $E_{Q'}$ coincide on R'-upsets, and it is straightforward to see by induction that the set $\{[y] \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}$ is an R'-upset. Therefore, by the induction hypothesis,

$$\begin{aligned}\mathfrak{F}^t, [x] \vDash_{v'} \exists\psi &\text{ iff } (\exists [y] \in X')([x]E_{Q'}[y] \ \&\ \mathfrak{F}^t, [y] \vDash_{v'} \psi)\\ &\text{ iff } [x] \in E_{Q'}[\{[y] \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}]\end{aligned}$$

$$\text{iff } [x] \in Q'[\{[y] \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}]$$

$$\text{iff } x \in Q_E[\{y \mid \mathfrak{F}^t, [y] \vDash_{v'} \psi\}]$$

$$\text{iff } x \in Q_E[\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}].$$

On the other hand,

$$\mathfrak{F}, x \vDash_v (\exists\psi)^t \text{ iff } \mathfrak{F}, x \vDash_v \exists\psi^t$$

$$\text{iff } (\exists y \in X)(xEy \ \& \ \mathfrak{F}, y \vDash_v \psi^t)$$

$$\text{iff } x \in E[\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}]$$

$$\text{iff } x \in Q_E[\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}]$$

since, by Lemma 2.29, the set $\{y \mid \mathfrak{F}, y \vDash_v \psi^t\}$ is an R-upset, and E and Q_E coincide on R-upsets. Thus, $\mathfrak{F}^t, [x] \vDash_{v'} \exists\psi$ iff $\mathfrak{F}, x \vDash_v (\exists\psi)^t$.

(3). If $\mathfrak{F} \nvDash \varphi^t$, then there is a valuation v on $\mathfrak{F}$ such that $\mathfrak{F}, x \nvDash_v \varphi^t$ for some $x \in X$. By (2), v' is a valuation on $\mathfrak{F}^t$ such that $\mathfrak{F}^t, [x] \nvDash_{v'} \varphi$. Therefore, $\mathfrak{F}^t \nvDash \varphi$. If $\mathfrak{F}^t \nvDash \varphi$, then there is a valuation w on $\mathfrak{F}^t$ and $[x] \in X'$ such that $\mathfrak{F}^t, [x] \nvDash_w \varphi$. Let v be the valuation on $\mathfrak{F}$ given by $v(p) = \{x \mid [x] \in w(p)\}$. Since $\mathfrak{F}^t$ is an MIPC-frame, $w(p)$ is an R'-upset of $\mathfrak{F}^t$ for each p. So $v(p)$ is an R-upset of $\mathfrak{F}$ for each p. Therefore, $w = v'$ because

$$v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\} = \{[x] \in X' \mid x \in v(p)\} = w(p).$$

Thus, $\mathfrak{F}^t, [x] \nvDash_{v'} \varphi$. By (2), $\mathfrak{F}, x \nvDash_v \varphi^t$. Consequently, $\mathfrak{F} \nvDash \varphi^t$.

(4). Let $\mathfrak{G} = (X, R, Q)$ be an MIPC-frame. We show that $\mathfrak{F} = (X, R, E_Q)$ is an $\mathsf{MS4}$-frame. If xE_Qy and yRz, then by definition of E_Q and condition (O1) of MIPC-frames, xQy and yQz. Since Q is transitive, xQz. Condition (O2) then implies that there is $u \in X$ with

xRu and uE_Qz. Thus, $\mathfrak{F}$ is an $\mathsf{MS4}$-frame. Since R is a partial order, $\sim$ is the identity relation. It then follows from condition (O2) that $Q = Q_{E_Q}$, and hence $\mathfrak{G}$ is isomorphic to $\mathfrak{F}^t$. □

Remark 2.32. In general, we cannot recover an $\mathsf{MS4}$-frame $\mathfrak{F} = (X, R, E)$ from its skeleton $\mathfrak{F}^t$ even if R is a partial order. Indeed, it is not always the case that $E = E_{Q_E}$. However, if $\mathfrak{F}$ is canonical (and in particular finite), then $E = E_{Q_E}$; see [11, Sec. 2] for details.

We are now ready to give an alternate proof of the fullness and faithfulness of the monadic Gödel translation.

Theorem 2.33. *The Gödel translation of* MIPC *into* $\mathsf{MS4}$ *is full and faithful; that is,*

$$\mathsf{MIPC} \vdash \varphi \quad \textit{iff} \quad \mathsf{MS4} \vdash \varphi^t.$$

Proof. To prove faithfulness, suppose that $\mathsf{MS4} \nvdash \varphi^t$. By Theorem 2.22, there is an $\mathsf{MS4}$-frame $\mathfrak{F}$ such that $\mathfrak{F} \nvDash \varphi^t$. By Proposition 2.31, $\mathfrak{F}^t$ is an MIPC-frame and $\mathfrak{F}^t \nvDash \varphi$. Thus, by Theorem 2.13, $\mathsf{MIPC} \nvdash \varphi$. For fullness, let $\mathsf{MIPC} \nvdash \varphi$. Then there is an MIPC frame $\mathfrak{G}$ such that $\mathfrak{G} \nvDash \varphi$. By Proposition 2.31(4), there is an $\mathsf{MS4}$-frame such that $\mathfrak{G}$ isomorphic to $\mathfrak{F}^t$. Therefore, $\mathfrak{F}^t \nvDash \varphi$. Proposition 2.31(3) implies that $\mathfrak{F} \nvDash \varphi^t$. Thus, $\mathsf{MS4} \nvdash \varphi^t$. □

3 Temporal interpretation of monadic intuitionistic quantifiers

The goal of this section is to provide a modification of the Gödel translation that realizes the temporal interpretation of monadic intuitionistic quantifiers as "always in the future" for $\forall$ and "sometime in the past" for $\exists$. We introduce a new tense logic $\mathsf{TS4}$ that will be the target of the translation. In order to define $\mathsf{TS4}$, it is convenient to first describe the tense logic $\mathsf{S4.t}$. We then define the temporal translation of MIPC into $\mathsf{TS4}$ and prove that it is full and faithful using relational semantics. We compare this new temporal translation with the standard Gödel translation of MIPC into $\mathsf{MS4}$ described in the previous section. For this, we introduce the logic $\mathsf{MS4.t}$ and show that both $\mathsf{MS4}$ and $\mathsf{TS4}$ can be translated fully and faithfully into $\mathsf{MS4.t}$. All these translations together form a diagram that is commutative up to logical equivalence. We end the section by proving that $\mathsf{MS4.t}$ has the finite model property (fmp). Since all the translations into $\mathsf{MS4.t}$ are full and faithful, as a consequence we obtain that the other logics involved also have the fmp.

3.1 $\mathsf{S4.t}$

The tense logic $\mathsf{S4.t}$ is the extension of the least tense logic $\mathsf{K.t}$ in which both tense modalities satisfy the $\mathsf{S4}$-axioms. This system was studied by several authors. In particular, Esakia [49] showed that an extension of the Gödel translation embeds the Heyting-Brouwer logic HB of Rauszer [101] into $\mathsf{S4.t}$ fully and faithfully. The language of HB is obtained by enriching the language of IPC by an additional connective of coimplication, and the logic HB is the extension of IPC by the axioms for coimplication, which are dual to the axioms for implication.

Wolter [113] extended the celebrated Blok-Esakia Theorem to this setting.

Let $\mathcal{L}_T$ be the propositional tense language with two modalities $\Box_F$ and $\Box_P$. As usual, $\Box_F$ is interpreted as "always in the future" and $\Box_P$ as "always in the past." We use the following standard abbreviations: $\Diamond_F$ for $\neg\Box_F\neg$ and $\Diamond_P$ for $\neg\Box_P\neg$. Then $\Diamond_F$ is interpreted as "sometime in the future" and $\Diamond_P$ as "sometime in the past."

Definition 3.1. Let $\mathsf{S4.t}$ be the smallest classical bimodal logic containing the $\mathsf{S4}$-axioms for $\Box_F$ and $\Box_P$, the tense axioms

$$p \to \Box_P\Diamond_F p$$

$$p \to \Box_F\Diamond_P p$$

and closed under modus ponens, substitution, $\Box_F$-necessitation, and $\Box_P$-necessitation.

3.1.1 $\mathsf{S4.t}$-algebras

Algebraic semantics for $\mathsf{S4.t}$ was studied by Esakia [49], where the duality theory for $\mathsf{S4}$-algebras was generalized to $\mathsf{S4.t}$-algebras.

Definition 3.2. An $\mathsf{S4.t}$-*algebra* is a triple $(B, \Box_F, \Box_P)$ where $(B, \Box_F)$, $(B, \Box_P)$ are $\mathsf{S4}$-algebras and for each $a \in B$ we have

$$a \leq \Box_P\Diamond_F a \tag{PF}$$

$$a \leq \Box_F\Diamond_P a \tag{FP}$$

The Lindenbaum-Tarski construction yields that $\mathsf{S4.t}$-algebras provide a sound and complete algebraic semantics for $\mathsf{S4.t}$.

3.1.2 Relational semantics

Relational semantics for $\mathsf{S4.t}$ is given by $\mathsf{S4.t}$-frames.

Definition 3.3. An $\mathsf{S4.t}$-*frame* is a pair $\mathfrak{F} = (X, R)$ where X is a set and R is a quasi-order on X.

A *valuation* on an $\mathsf{S4.t}$-frame $\mathfrak{F} = (X, R)$ is a map v associating a subset of X to each propositional letter of $\mathcal{L}_T$. The classical connectives are interpreted as usual, and the tense modalities are interpreted as

$$\begin{array}{lll} x \vDash_v \Box_F \varphi & \text{iff} & (\forall y \in X)(xRy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v \Box_P \varphi & \text{iff} & (\forall y \in X)(yRx \Rightarrow y \vDash_v \varphi). \end{array}$$

As usual, we say that φ is *valid* in $\mathfrak{F}$, in symbols $\mathfrak{F} \vDash \varphi$, if $x \vDash_v \varphi$ for every valuation v and $x \in X$.

It is straightforward to see that all the axioms of $\mathsf{S4.t}$ are Sahlqvist formulas. Therefore, by the Sahlqvist completeness theorem we have that $\mathsf{S4.t}$ is canonical and is complete with respect to the relational semantics. That $\mathsf{S4.t}$ has the fmp follows from [103, pp. 313–314] (see also [67, p. 44] and Remark 3.44).

We also have the following representation of $\mathsf{S4.t}$-algebras. Let $R^{\smile}$ be the converse of R. For $U \in \wp(X)$ let

$$\Box_R(U) = X \setminus R^{-1}[X \setminus U] \text{ and } \Box_{R^{\smile}}(U) = X \setminus R[X \setminus U].$$

Since R is a quasi-order, so is $R^{\smile}$, so $(\wp(X), \Box_R)$ and $(\wp(X), \Box_{R^{\smile}})$ are $\mathsf{S4}$-algebras. A standard argument (see [77, Thm. 3.6]) gives that $\mathfrak{F}^+ := (\wp(X), \Box_R, \Box_{R^{\smile}})$ satisfies (PF) and (FP). Therefore, $\mathfrak{F}^+$ is an $\mathsf{S4.t}$-algebra, and each $\mathsf{S4.t}$-algebra $\mathfrak{B} = (B, \Box_F, \Box_P)$ is isomorphic to a subalgebra of $\mathfrak{F}^+$ for some $\mathsf{S4.t}$-frame $\mathfrak{F}$. As usual, we can take $\mathfrak{F}$ to be the canonical frame of $\mathfrak{B}$. Let H_F and H_P be the sets of $\Box_F$-fixpoints and $\Box_P$-fixpoints, respectively. Since $\Box_F$ and $\Box_P$ are S4-operators, H_F and H_P are Heyting algebras.

Remark 3.4. Let $(B, \Box_F, \Box_P)$ be an S4.t-algebra. It follows from Definition 3.2 that H_F coincides with the set of $\Diamond_P$-fixpoints and H_P with the set of $\Diamond_F$-fixpoints. Moreover, $\neg$ maps H_F to H_P and vice versa. Indeed, if $a \in H_F$, then $a = \Box_F a$. By (PF), $\Diamond_P a = \Diamond_P \Box_F a \leq a$, so $\Diamond_P a = a$, and hence $\Box_P \neg a = \neg \Diamond_P a = \neg a$. Therefore, $\neg a \in H_P$. Similarly, if $a \in H_P$, then $\neg a \in H_F$. Thus, $\neg$ is a dual isomorphism between H_F and H_P.

Let $\mathfrak{B} = (B, \Box_F, \Box_P)$ be an S4.t-algebra. The *canonical frame* of $\mathfrak{B}$ is the frame $\mathfrak{B}_+ = (X_\mathfrak{B}, R_\mathfrak{B})$ where $X_\mathfrak{B}$ is the set of ultrafilters of B and $x R_\mathfrak{B} y$ iff $x \cap H_F \subseteq y$; equivalently, $y \cap H_P \subseteq x$. By a standard argument, if $\mathfrak{B}$ is an S4.t-algebra, then $\mathfrak{B}_+$ is an S4.t-frame and we have the following representation theorem:

Proposition 3.5. *If $\mathfrak{B}$ is an* S4.t*-algebra, then $\mathfrak{B}$ is isomorphic to a subalgebra of* $(\mathfrak{B}_+)^+$.

Remark 3.6. To recover the image of $\mathfrak{B}$ in $\wp(X_\mathfrak{B})$ we need to endow $X_\mathfrak{B}$ with a Stone topology. This leads to the notion of *perfect* S4.t*-frames* and a duality between the category of S4.t-algebras and the category of perfect S4.t-frames (see [49]). When $\mathfrak{B}$ is finite, its embedding into $(\mathfrak{B}_+)^+$ is an isomorphism, and hence the categories of finite S4.t-algebras and finite S4.t-frames are dually equivalent.

3.2 TS4

The tense logic TS4 will combine S4 with S4.t. We will use S4 to interpret intuitionistic connectives, and S4.t to interpret monadic intuitionistic quantifiers. Let $\mathcal{ML}$ be the multi-modal propositional language with three modalities $\Box$, $\blacksquare_F$, and $\blacksquare_P$. We use $\Diamond$, $\blacklozenge_F$, and $\blacklozenge_P$

as usual abbreviations.

Definition 3.7. The logic TS4 is the least classical multimodal logic containing the S4-axioms for $\Box$, $\blacksquare_F$, and $\blacksquare_P$, the tense axioms for $\blacksquare_F$ and $\blacksquare_P$, the connecting axioms

$$\Diamond p \to \blacklozenge_F p$$

$$\blacklozenge_F p \to \Diamond(\blacklozenge_F p \wedge \blacklozenge_P p)$$

and closed under modus ponens, substitution, and three necessitation rules (for $\Box$, $\blacksquare_F$, and $\blacksquare_P$).

3.2.1 TS4-algebras

Algebraic semantics for TS4 is given by TS4-algebras.

Definition 3.8. A TS4-*algebra* is a quadruple $\mathfrak{B} = (B, \Box, \blacksquare_F, \blacksquare_P)$ where $(B, \Box)$ is an S4-algebra, $(B, \blacksquare_F, \blacksquare_P)$ is an S4.t-algebra, and for each $a \in B$ we have:

$$\Diamond a \leq \blacklozenge_F a \tag{T1}$$

$$\blacklozenge_F a \leq \Diamond(\blacklozenge_F a \wedge \blacklozenge_P a) \tag{T2}$$

The Lindenbaum-Tarski construction then yields that TS4-algebras provide a sound and complete algebraic semantic for TS4.

3.2.2 Relational semantics

Definition 3.9. A TS4-*frame* is a triple $\mathfrak{F} = (X, R, Q)$ where X is a set and R, Q are quasi-orders on X such that $R \subseteq Q$ and xQy implies that there is $z \in X$ such that xRz and zE_Qy.

Remark 3.10.

1. The only difference between $\mathsf{TS4}$-frames and MIPC-frames is that in $\mathsf{TS4}$-frames the relation R is a quasi-order, while in MIPC-frames it is a partial order.

2. It is straightforward to check that if (X, R, Q) is a $\mathsf{TS4}$-frame, then (X, R, E_Q) is an $\mathsf{MS4}$-frame, and that if (X, R, E) is an $\mathsf{MS4}$-frame, then (X, R, Q_E) is a $\mathsf{TS4}$-frame (see Definition 2.30). If (X, R, Q) is a $\mathsf{TS4}$-frame, by definition we have that xQy iff $(\exists z \in X)(xRz \ \& \ zE_Q y)$. Thus, $Q = Q_{E_Q}$. On the other hand, there exist $\mathsf{MS4}$-frames (X, R, E) such that $E \neq E_{Q_E}$ (see [11, p. 24]). Therefore, this correspondence is not a bijection.

A *valuation* of $\mathcal{ML}$ into a $\mathsf{TS4}$-frame $\mathfrak{F} = (X, R, Q)$ associates with each propositional letter a subset of X. The classical connectives are interpreted as usual, $\Box$ is interpreted using the relation R, and $\blacksquare_F$, $\blacksquare_P$ are interpreted using the relation Q:

$$\begin{array}{lll} x \vDash_v \Box\varphi & \text{iff} & (\forall y \in X)(xRy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v \blacksquare_F\varphi & \text{iff} & (\forall y \in X)(xQy \Rightarrow y \vDash_v \varphi), \\ x \vDash_v \blacksquare_P\varphi & \text{iff} & (\forall y \in X)(yQx \Rightarrow y \vDash_v \varphi). \end{array}$$

Consequently,

$$\begin{array}{lll} x \vDash_v \Diamond\varphi & \text{iff} & (\exists y \in X)(xRy \ \& \ y \vDash_v \varphi), \\ x \vDash_v \blacklozenge_F\varphi & \text{iff} & (\exists y \in X)(xQy \ \& \ y \vDash_v \varphi), \\ x \vDash_v \blacklozenge_P\varphi & \text{iff} & (\exists y \in X)(yQx \ \& \ y \vDash_v \varphi). \end{array}$$

All the axioms of $\mathsf{TS4}$ are Sahlqvist formulas. Therefore, by the Sahlqvist completeness theorem we have:

Theorem 3.11. $\mathsf{TS4}$ *is canonical and hence is complete with respect to the relational semantics, i.e.*

$$\mathsf{TS4} \vdash \varphi \ \textit{ iff } \ \mathfrak{F} \vDash \varphi \textit{ for every } \mathsf{TS4}\textit{-frame } \mathfrak{F}.$$

In Section 3.5 we will prove that TS4 has the fmp and hence is decidable. We conclude this section by proving a representation theorem for TS4-algebras.

Lemma 3.12. *If* $\mathfrak{F} = (X, R, Q)$ *is a* TS4*-frame, then* $\mathfrak{F}^+ = (\wp(X), \Box_R, \Box_Q, \Box_{Q^\smile})$ *is a* TS4*-algebra.*

Proof. Since R and Q are quasi-orders, $(\wp(X), \Box_R)$ is an S4-algebra and $(\wp(X), \Box_Q, \Box_{Q^\smile})$ is an S4.t-algebra. It remains to show that $\mathfrak{F}^+$ satisfies (T1) and (T2).

(T1) Since $R \subseteq Q$, we have $\Diamond_R(U) = R^{-1}[U] \subseteq Q^{-1}[U] = \Diamond_Q(U)$.

(T2) Let $x \in \Diamond_Q(U) = Q^{-1}[U]$, so there is $y \in U$ with xQy. Then there is $z \in X$ with xRz and zE_Qy. Therefore, $z \in Q^{-1}[y] \subseteq Q^{-1}[U] = \Diamond_Q(U)$ and $z \in Q[y] \subseteq Q[U] = \Diamond_{Q^\smile}(U)$. Thus, $x \in R^{-1}[z] \subseteq R^{-1}[\Diamond_Q(U) \cap \Diamond_{Q^\smile}(U)] = \Diamond_R(\Diamond_Q(U) \cap \Diamond_{Q^\smile}(U))$. This shows that $\Diamond_Q(U) \subseteq \Diamond_R(\Diamond_Q(U) \cap \Diamond_{Q^\smile}(U))$.

□

We next prove that each TS4-algebra is represented as a subalgebra of $\mathfrak{F}^+$ for some TS4-frame $\mathfrak{F}$. For a TS4-algebra $(B, \Box, \blacksquare_F, \blacksquare_P)$ let H, H_F, and H_P be the Heyting algebras of the $\Box$-fixpoints, $\blacksquare_F$-fixpoints, and $\blacksquare_P$-fixpoints, respectively.

Definition 3.13. Let $\mathfrak{B} = (B, \Box, \blacksquare_F, \blacksquare_P)$ be a TS4-algebra. The *canonical frame* of $\mathfrak{B}$ is the frame $\mathfrak{B}_+ = (X_\mathfrak{B}, R_\mathfrak{B}, Q_\mathfrak{B})$ where $X_\mathfrak{B}$ is the set of ultrafilters of B, $xR_\mathfrak{B}y$ iff $x \cap H \subseteq y$, and $xQ_\mathfrak{B}y$ iff $x \cap H_F \subseteq y$, which happens iff $y \cap H_P \subseteq x$.

Lemma 3.14. *If* $\mathfrak{B}$ *is a* TS4*-algebra, then* $\mathfrak{B}_+$ *is a* TS4*-frame.*

Proof. Clearly $R_\mathfrak{B}$ and $Q_\mathfrak{B}$ are quasi-orders. To prove that $R_\mathfrak{B} \subseteq Q_\mathfrak{B}$ we first show that $H_F \subseteq H$. Let $a \in H_F$. Then $a = \blacksquare_F a = \neg\blacklozenge_F\neg a = \neg\blacklozenge_F\blacklozenge_F\neg a$. By (T1),

$$\neg\blacklozenge_F\blacklozenge_F\neg a \leq \neg\Diamond\blacklozenge_F\neg a = \Box\blacksquare_F a \leq \Box a.$$

Therefore, $a = \Box a$, and so $a \in H$. Now suppose that $xR_\mathfrak{B}y$, so $x \cap H \subseteq y$. Let $a \in x \cap H_F$. Then $a \in x \cap H \subseteq y$. Thus, $a \in y$, and hence $xQ_\mathfrak{B}y$.

To prove the other condition, let $xQ_\mathfrak{B}y$, so $x \cap H_F \subseteq y$. We show that the subset $(x \cap H) \cup (y \cap H_F) \cup (y \cap H_P)$ generates a proper filter of B. Otherwise, since H, H_F, H_P are closed under meets, there are $a \in x \cap H$, $b \in y \cap H_F$, and $c \in y \cap H_P$ such that $a \wedge b \wedge c = 0$. By Remark 3.4, H_F coincides with the set of $\blacklozenge_P$-fixpoints and H_P with the set of $\blacklozenge_F$-fixpoints. Therefore, since $b \in H_F$ and $c \in H_P$, we have $\blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c) \leq \blacklozenge_P b \wedge \blacklozenge_F c = b \wedge c$. Thus, $a \wedge \blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c) \leq a \wedge b \wedge c = 0$, yielding $a \leq \neg(\blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c))$. Since $a \in H$, we have

$$a = \Box a \leq \Box\neg(\blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c)) = \neg\Diamond(\blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c)).$$

Consequently, $a \wedge \Diamond(\blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c)) = 0$. By (T2),

$$a \wedge \blacklozenge_F(b \wedge c) \leq a \wedge \Diamond(\blacklozenge_P(b \wedge c) \wedge \blacklozenge_F(b \wedge c)) = 0.$$

Because $b \wedge c \leq \blacklozenge_F(b \wedge c)$, $b \wedge c \in y$, and y is a filter, we have $\blacklozenge_F(b \wedge c) \in y$. Since $x \cap H_F \subseteq y$, we have $y \cap H_P \subseteq x$. Therefore, $\blacklozenge_F(b \wedge c) \in y \cap H_P \subseteq x$ and $a \in x$. Thus, $0 = a \wedge \blacklozenge_F(b \wedge c) \in x$, a contradiction. Consequently, there is an ultrafilter z such that $(x \cap H) \cup (y \cap H_F) \cup (y \cap H_P) \subseteq z$. But then $x \cap H \subseteq z$, $y \cap H_F \subseteq z$, and $y \cap H_P \subseteq z$. This gives that $xR_\mathfrak{B}z$, $yQ_\mathfrak{B}z$, and $zQ_\mathfrak{B}y$, as desired. $\Box$

Definition 3.15. We call a $\mathsf{TS4}$-frame *canonical* if it is isomorphic to $\mathfrak{B}_+$ for some $\mathsf{TS4}$-algebra $\mathfrak{B}$.

Let $\mathfrak{B}$ be a $\mathsf{TS4}$-algebra. Since $\beta : B \to \wp(X_\mathfrak{B})$ is an embedding of $\mathsf{TS4}$-algebras, we obtain the following representation theorem for $\mathsf{TS4}$-algebras.

Proposition 3.16. *Each* $\mathsf{TS4}$-*algebra* $\mathfrak{B}$ *is isomorphic to a subalgebra of* $(\mathfrak{B}_+)^+$.

Remark 3.17. To recover the image of $\mathfrak{B}$ in $\wp(X_\mathfrak{B})$ we need to endow $X_\mathfrak{B}$ with a Stone topology. This leads to the notion of *perfect* $\mathsf{TS4}$-*frames* and a duality between the categories of $\mathsf{TS4}$-algebras and perfect $\mathsf{TS4}$-frames which generalizes Esakia duality for $\mathsf{S4.t}$. When $\mathfrak{B}$ is finite, its embedding into $(\mathfrak{B}_+)^+$ is an isomorphism, and hence the categories of finite $\mathsf{TS4}$-algebras and finite $\mathsf{TS4}$-frames are dually equivalent.

3.3 Temporal translation of MIPC into TS4

We now modify the Gödel translation in order to obtain a full and faithful translation of MIPC into $\mathsf{TS4}$ that realizes the desired temporal interpretation of the monadic intuitionistic quantifiers.

Definition 3.18. The translation $(-)^\natural : \mathsf{MIPC} \to \mathsf{TS4}$ is defined as $(-)^t$ on propositional letters, $\bot$, $\wedge$, $\vee$, and $\to$; and for $\forall$ and $\exists$ we set:

$$(\forall\varphi)^\natural = \blacksquare_F \varphi^\natural$$

$$(\exists\varphi)^\natural = \blacklozenge_P \varphi^\natural$$

Thus, $\forall$ is interpreted as "always in the future" and $\exists$ as "sometime in the past."

We adapt Definition 2.30 to the setting of $\mathsf{TS4}$-frames by utilizing the correspondence between $\mathsf{TS4}$-frames and $\mathsf{MS4}$-frames described in Remark 3.10.

Definition 3.19. Let $\mathfrak{F} = (X, R, Q)$ be a $\mathsf{TS4}$-frame, and let $\sim$ be the equivalence relation given by $x \sim y$ iff xRy and yRx. We set X' to be the set of equivalence classes of $\sim$, and define R' and Q' on X' by $[x]R'[y]$ iff xRy and $[x]Q'[y]$ iff xQy. We call $\mathfrak{F}^\natural = (X', R', Q')$ the *skeleton* of $\mathfrak{F}$.

Proposition 3.20.

1. *If $\mathfrak{F}$ is a* $\mathsf{TS4}$*-frame, then $\mathfrak{F}^\natural$ is an* MIPC*-frame.*

2. *For each valuation v on $\mathfrak{F}$ there is a valuation v' on $\mathfrak{F}^\natural$ such that for each $x \in \mathfrak{F}$ and $\mathcal{L}_{\forall\exists}$-formula φ, we have*

$$\mathfrak{F}^\natural, [x] \vDash_{v'} \varphi \textit{ iff } \mathfrak{F}, x \vDash_v \varphi^\natural.$$

3. *For each $\mathcal{L}_{\forall\exists}$-formula φ, we have*

$$\mathfrak{F}^\natural \vDash \varphi \textit{ iff } \mathfrak{F} \vDash \varphi^\natural.$$

4. *Any* MIPC*-frame $\mathfrak{G}$ is also a* $\mathsf{TS4}$*-frame and $\mathfrak{G}^\natural$ is isomorphic to $\mathfrak{G}$.*

Proof. (1). It is well known that (X', R') is an intuitionistic Kripke frame. The relation Q' is well defined on X' because $R \subseteq Q$ in $\mathfrak{F}$. Showing that Q' is a quasi-order, and that (O1) and (O2) hold in $\mathfrak{F}^\natural$ is straightforward.

(2). As in Proposition 2.31(2), we define v' by $v'(p) = \{[x] \in X' \mid R[x] \subseteq v(p)\}$ and show that $\mathfrak{F}^\natural, [x] \vDash_{v'} \varphi$ iff $\mathfrak{F}, x \vDash_v \varphi^\natural$ by induction on the complexity of φ. It is sufficient to only consider the cases when φ is of the form $\forall\psi$ or $\exists\psi$. Suppose $\varphi = \forall\psi$. Then by the definition of Q' and induction hypothesis,

$$\mathfrak{F}^\natural, [x] \vDash_{v'} \forall\psi \text{ iff } (\forall [y] \in X')([x]Q'[y] \Rightarrow \mathfrak{F}^\natural, [y] \vDash_{v'} \psi)$$

$$\text{iff } (\forall y \in X)(xQy \Rightarrow \mathfrak{F}^\natural, [y] \vDash_{v'} \psi)$$

$$\text{iff } (\forall y \in X)(xQy \Rightarrow \mathfrak{F}, y \vDash_v \psi^\natural)$$

$$\text{iff } \mathfrak{F}, x \vDash_v \blacksquare_F \psi^\natural$$

$$\text{iff } \mathfrak{F}, x \vDash_v (\forall\psi)^\natural.$$

Suppose $\varphi = \exists\psi$. As noted in Remark 2.7, Q' and $E_{Q'}$ coincide on R'-upsets. Since the set $\{[y] \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}$ is an R'-upset, by the induction hypothesis, we have

$$\mathfrak{F}^\natural, [x] \vDash_{v'} \exists\psi \text{ iff } (\exists [y] \in X')([x]E_{Q'}[y] \;\&\; \mathfrak{F}^\natural, [y] \vDash_{v'} \psi)$$

$$\text{iff } [x] \in E_{Q'}[\{[y] \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}]$$

$$\text{iff } [x] \in Q'[\{[y] \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}]$$

$$\text{iff } x \in Q[\{y \mid \mathfrak{F}^\natural, [y] \vDash_{v'} \psi\}]$$

$$\text{iff } x \in Q[\{y \mid \mathfrak{F}, y \vDash_v \psi^\natural\}]$$

$$\text{iff } (\exists y \in X)(yQx \;\&\; \mathfrak{F}, y \vDash_v \psi^\natural)$$

$$\text{iff } \mathfrak{F}, x \vDash_v \blacklozenge_P \psi^\natural$$

$$\text{iff } \mathfrak{F}, x \vDash_v (\exists\psi)^\natural.$$

(3). The proof is analogous to that of Proposition 2.31(3).

(4). Let $\mathfrak{G} = (X, R, Q)$ be an MIPC-frame. It is clear from the definition of TS4-frames that $\mathfrak{G}$ is also a TS4-frame. Since R is a partial order, $\sim$ is the identity relation. Therefore, $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^\natural$. □

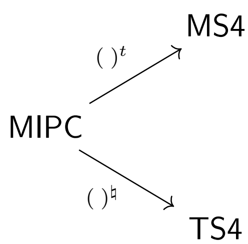
Theorem 3.21. *The translation $(-)^\natural$ of* MIPC *into* TS4 *is full and faithful; that is,*

$$\mathsf{MIPC} \vdash \varphi \textit{ iff } \mathsf{TS4} \vdash \varphi^\natural.$$

Proof. To prove faithfulness, suppose that $\mathsf{TS4} \nvdash \varphi^\natural$. By Theorem 3.11, there is a $\mathsf{TS4}$-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi^\natural$. By Proposition 3.20, $\mathfrak{F}^\natural$ is an MIPC-frame and $\mathfrak{F}^\natural \not\models \varphi$. Thus, by Theorem 2.13, $\mathsf{MIPC} \nvdash \varphi$. For fullness, if $\mathsf{MIPC} \nvdash \varphi$, then there is an MIPC-frame $\mathfrak{G}$ such that $\mathfrak{G} \not\models \varphi$. By Proposition 3.20(4), $\mathfrak{G}$ is also a $\mathsf{TS4}$-frame and it is isomorphic to $\mathfrak{G}^\natural$. Therefore, $\mathfrak{G}^\natural \not\models \varphi$. Proposition 3.20(3) then yields that $\mathfrak{G} \not\models \varphi^\natural$. Thus, $\mathsf{TS4} \nvdash \varphi^\natural$. □

3.4 Translations into MS4.t

In Sections 2.3 and 3.3 we described full and faithful translations of MIPC into MS4 and TS4, respectively. This yields the following diagram.



There does not appear to be a natural way to translate MS4 into TS4 or vice versa. The aim of this section is to define a new tense system and show that both MS4 and TS4 embed fully and faithfully into it, thus completing the above diagram.

3.4.1 MS4.t

Let $\mathcal{L}_{T\forall}$ be the propositional language with the tense modalities $\Box_F$ and $\Box_P$, and the monadic modality $\forall$. In order to stress that the language $\mathcal{L}_{T\forall}$ is different from $\mathcal{ML}$ and TS4, we use different symbols for the tense modalities.

Definition 3.22. The *tense* MS4, denoted MS4.t, is the least classical multimodal logic containing the S4.t-axioms for $\Box_F$ and $\Box_P$, the S5-axioms for $\forall$, the left commutativity axiom

$$\Box_F \forall p \to \forall \Box_F p,$$

and closed under modus ponens, substitution, and the necessitation rules (for $\Box_F$, $\Box_P$, and $\forall$).

Algebraic semantics for MS4.t is given by MS4.t-algebras.

Definition 3.23. An MS4.t-*algebra* is a tuple $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ where $(B, \Box_F, \Box_P)$ is an S4.t-algebra and $(B, \Box_F, \forall)$ is an MS4-algebra.

As usual, the Lindenbaum-Tarski construction yields that MS4.t is sound and complete with respect to MS4.t-algebras.

As with S4 and S4.t, we have that MS4.t-frames are simply MS4-frames. A *valuation* on an MS4.t-frame $\mathfrak{F} = (X, R, E)$ is a map v associating to each propositional letter of $\mathcal{L}_{T\forall}$ a subset of $\mathfrak{F}$. The boolean connectives are interpreted as usual, and

$$\begin{array}{lll} \mathfrak{F}, x \vDash_v \Box_F \varphi & \text{iff} & (\forall y \in X)(xRy \Rightarrow y \vDash_v \varphi), \\ \mathfrak{F}, x \vDash_v \Box_P \varphi & \text{iff} & (\forall y \in X)(yRx \Rightarrow y \vDash_v \varphi), \\ \mathfrak{F}, x \vDash_v \forall \varphi & \text{iff} & (\forall y \in X)(xEy \Rightarrow y \vDash_v \varphi). \end{array}$$

Since both MS4 and S4.t can be axiomatized by Sahlqvist formulas, this is also true for MS4.t. Therefore, we have:

Theorem 3.24. MS4.t *is canonical and hence is complete with respect to the relational semantics, i.e.*

$$\mathsf{MS4.t} \vdash \varphi \ \textit{ iff } \ \mathfrak{F} \vDash \varphi \textit{ for every } \mathsf{MS4.t}\textit{-frame } \mathfrak{F}.$$

In Section 3.5 we will prove that MS4.t has the fmp and hence is decidable. We conclude this section by proving a representation theorem for MS4.t-algebras. The following lemma is an immediate consequence of the fact that MS4.t-frames are the same as MS4-frames.

Lemma 3.25. *If $\mathfrak{F} = (X, R, E)$ is an* MS4.t*-frame, then $\mathfrak{F}^+ := (\wp(X), \Box_R, \Box_{R^\smile}, \forall_E)$ is an* MS4.t*-algebra.*

We next prove that each MS4.t-algebra is represented as a subalgebra of $\mathfrak{F}^+$ for some MS4.t-frame $\mathfrak{F}$. For an MS4.t-algebra $(B, \Box_F, \Box_P, \forall)$ let H_F, H_P, and B_0 be the $\Box_F$-fixpoints, $\Box_P$-fixpoints, and $\forall$-fixpoints, respectively. Clearly H_F and H_P are Heyting algebras and B_0 is a boolean subalgebra of B.

Definition 3.26. Let $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ be an MS4.t-algebra. The *canonical frame* of $\mathfrak{B}$ is the frame $\mathfrak{B}_+ = (X_\mathfrak{B}, R_\mathfrak{B}, E_\mathfrak{B})$ where $X_\mathfrak{B}$ is the set of ultrafilters of B, $xR_\mathfrak{B}y$ iff $x \cap H_F \subseteq y$ iff $y \cap H_P \subseteq x$, and $xE_\mathfrak{B}y$ iff $x \cap B_0 = y \cap B_0$.

Since MS4.t-frames are MS4-frames, the next lemma is obvious.

Lemma 3.27. *If $\mathfrak{B}$ is an* MS4.t*-algebra, then $\mathfrak{B}_+$ is an* MS4.t*-frame.*

Thus, since $\beta : B \to \wp(X_\mathfrak{B})$ is an embedding of S4.t-algebras and MS4-algebras, we obtain the following representation theorem for MS4.t-algebras.

Proposition 3.28. *Each* MS4.t*-algebra $\mathfrak{B}$ is isomorphic to a subalgebra of $(\mathfrak{B}_+)^+$.*

Remark 3.29. To recover the image of the embedding of $\mathfrak{B}$ into $(\mathfrak{B}_+)^+$ we need to endow $\mathfrak{B}_+$ with a Stone topology. This leads to the notion of *perfect* MS4.t-*frames* and a duality between the categories of MS4.t-algebras and perfect MS4.t-frames. When $\mathfrak{B}$ is finite, its embedding into $(\mathfrak{B}_+)^+$ is an isomorphism, and hence the categories of finite MS4.t-algebras and finite MS4.t-frames are dually equivalent.

3.4.2 Translations of TS4 and MS4 into MS4.t

We next define two full and faithful translations $(-)^{\#} : \mathsf{MS4} \to \mathsf{MS4.t}$ and $(-)^{\dagger} : \mathsf{TS4} \to \mathsf{MS4.t}$. The translation of MS4 into MS4.t will reflect that MS4.t is the tense extension of MS4.

Definition 3.30. We define the translation $(-)^{\#} : \mathsf{MS4} \to \mathsf{MS4.t}$ by replacing in each formula φ of $\mathcal{L}_{\Box\forall}$ every occurrence of $\Box$ with $\Box_F$.

Theorem 3.31. *The translation $(-)^{\#}$ of MS4 into MS4.t is full and faithful; that is,*

$$\mathsf{MS4} \vdash \varphi \quad \textit{iff} \quad \mathsf{MS4.t} \vdash \varphi^{\#}.$$

Proof. By definition, MS4.t-frames are MS4-frames and valuations on MS4-frames and MS4.t-frames coincide. The boolean connectives and monadic modality $\forall$ are interpreted the same way in MS4-frames and MS4.t-frames. Also, the interpretation of $\Box$ in MS4-frames coincides with the interpretation of $\Box_F$ in MS4.t-frames. This implies that for each frame $\mathfrak{F} = (X, R, E)$, valuation v, and $x \in X$, we have $\mathfrak{F}, x \vDash \varphi$ iff $\mathfrak{F}, x \vDash \varphi^{\#}$ for every $\mathcal{L}_{\Box\forall}$-formula φ. The result then follows from the soundness and completeness of MS4 and MS4.t with respect to their relational semantics (see Theorems 2.22 and 3.24). $\Box$

Definition 3.32. Define the translation $(-)^{\dagger} : \mathsf{TS4} \to \mathsf{MS4.t}$ by

$$p^{\dagger} = p \quad \text{for each propositional letter } p$$

$$(-)^{\dagger} \text{ commutes with the boolean connectives}$$

$$(\Box\varphi)^{\dagger} = \Box_F \varphi^{\dagger}$$

$$(\blacksquare_F \varphi)^{\dagger} = \Box_F \forall \varphi^{\dagger}$$

$$(\blacksquare_P \varphi)^{\dagger} = \forall \Box_P \varphi^{\dagger}.$$

Definition 3.33. For an $\mathsf{MS4.t}$-frame $\mathfrak{F} = (X, R, E)$ we define $\mathfrak{F}^\dagger = (X, R, Q_E)$.

Proposition 3.34.

1. *If $\mathfrak{F}$ is an $\mathsf{MS4.t}$-frame, then $\mathfrak{F}^\dagger$ is a $\mathsf{TS4}$-frame.*

2. *Each valuation v on $\mathfrak{F}$ is also a valuation on $\mathfrak{F}^\dagger$ such that for each $x \in \mathfrak{F}$ and $\mathcal{ML}$-formula φ, we have*
$$\mathfrak{F}^\dagger, x \vDash_v \varphi \textit{ iff } \mathfrak{F}, x \vDash_v \varphi^\dagger.$$

3. *For each $\mathcal{ML}$-formula φ, we have*
$$\mathfrak{F}^\dagger \vDash \varphi \textit{ iff } \mathfrak{F} \vDash \varphi^\dagger.$$

4. *For any $\mathsf{TS4}$-frame $\mathfrak{G}$ there is an $\mathsf{MS4.t}$-frame $\mathfrak{F}$ such that $\mathfrak{G} = \mathfrak{F}^\dagger$.*

Proof. (1). Since $\mathsf{MS4.t}$-frames coincide with $\mathsf{MS4}$-frames, we already observed in Remark 3.10(2) that $\mathfrak{F}^\dagger$ is a $\mathsf{TS4}$-frame.

(2). It is clear that if v is a valuation on $\mathfrak{F}$, then v is also a valuation on $\mathfrak{F}^\dagger$. We show that $\mathfrak{F}^\dagger, x \vDash_v \varphi$ iff $\mathfrak{F}, x \vDash_v \varphi^\dagger$ by induction on the complexity of φ. The only nontrivial cases are when φ is of the form $\Box\psi$, $\blacksquare_F\psi$ and $\blacksquare_P\psi$. Suppose $\varphi = \Box\psi$. Then, by the induction hypothesis,

$$\begin{aligned}\mathfrak{F}^\dagger, x \vDash_v \Box\psi &\text{ iff } (\forall y \in X)(xRy \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi)\\ &\text{ iff } (\forall y \in X)(xRy \Rightarrow \mathfrak{F}, y \vDash_v \psi^\dagger)\\ &\text{ iff } \mathfrak{F}, x \vDash_v \Box_F\psi^\dagger\\ &\text{ iff } \mathfrak{F}, x \vDash_v (\Box\psi)^\dagger.\end{aligned}$$

Suppose $\varphi = \blacksquare_F\psi$. Then, by the induction hypothesis,

$$\begin{aligned}
\mathfrak{F}^\dagger, x \vDash_v \blacksquare_F\psi &\text{ iff } (\forall y \in X)(xQ_Ey \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi)\\
&\text{ iff } (\forall z \in X)(xRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi))\\
&\text{ iff } (\forall z \in X)(xRz \Rightarrow (\forall y \in X)(zEy \Rightarrow \mathfrak{F}, y \vDash_v \psi^\dagger))\\
&\text{ iff } (\forall z \in X)(xRz \Rightarrow \mathfrak{F}, z \vDash \forall\psi^\dagger)\\
&\text{ iff } \mathfrak{F}, x \vDash_v \Box_F\forall\psi^\dagger\\
&\text{ iff } \mathfrak{F}, x \vDash_v (\blacksquare_F\psi)^\dagger.
\end{aligned}$$

Suppose $\varphi = \blacksquare_P\psi$. Then, by the induction hypothesis,

$$\begin{aligned}
\mathfrak{F}^\dagger, x \vDash_v \blacksquare_P\psi &\text{ iff } (\forall y \in X)(yQ_Ex \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi)\\
&\text{ iff } (\forall y, z \in X)(yRz \;\&\; zEx \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi)\\
&\text{ iff } (\forall z \in X)(zEx \Rightarrow (\forall y \in X)(yRz \Rightarrow \mathfrak{F}^\dagger, y \vDash_v \psi))\\
&\text{ iff } (\forall z \in X)(zEx \Rightarrow (\forall y \in X)(yRz \Rightarrow \mathfrak{F}, y \vDash_v \psi^\dagger))\\
&\text{ iff } (\forall z \in X)(zEx \Rightarrow \mathfrak{F}, z \vDash \Box_P\psi^\dagger)\\
&\text{ iff } (\forall z \in X)(xEz \Rightarrow \mathfrak{F}, z \vDash \Box_P\psi^\dagger)\\
&\text{ iff } \mathfrak{F}, x \vDash_v \forall\Box_P\psi^\dagger\\
&\text{ iff } \mathfrak{F}, x \vDash_v (\blacksquare_P\psi)^\dagger.
\end{aligned}$$

(3). The proof that $\mathfrak{F}^\dagger \vDash \varphi$ iff $\mathfrak{F} \vDash \varphi^\dagger$ is analogous to that of Proposition 2.31(3).

(4). Let $\mathfrak{G} = (X, R, Q)$ be a $\mathsf{TS4}$-frame. As we observed in Remark 3.10, $\mathfrak{F} = (X, R, E_Q)$ is an $\mathsf{MS4}$-frame, and so an $\mathsf{MS4.t}$-frame. By definition of $\mathsf{TS4}$-frames we have that $Q = Q_{E_Q}$, and hence $\mathfrak{G} = \mathfrak{F}^\dagger$. $\Box$

Theorem 3.35. *The translation* $(-)^\dagger$ *of* TS4 *into* MS4.t *is full and faithful; that is,*

$$\mathsf{TS4} \vdash \varphi \quad \textit{iff} \quad \mathsf{MS4.t} \vdash \varphi^\dagger.$$

Proof. To prove faithfulness, suppose that $\mathsf{MS4.t} \nvdash \varphi^\dagger$. By Theorem 3.24, there is an MS4.t-frame $\mathfrak{F}$ such that $\mathfrak{F} \nvDash \varphi^\dagger$. By Proposition 3.34, $\mathfrak{F}^\dagger$ is a TS4-frame and $\mathfrak{F}^\dagger \nvDash \varphi$. Thus, $\mathsf{TS4} \nvdash \varphi$ by Theorem 3.11. For fullness, if $\mathsf{TS4} \nvdash \varphi$, then there is a TS4-frame $\mathfrak{G}$ such that $\mathfrak{G} \nvDash \varphi$. By Proposition 3.34(4), there is an MS4.t-frame $\mathfrak{F}$ such that $\mathfrak{G}$ is isomorphic to $\mathfrak{F}^\dagger$. Therefore, $\mathfrak{F}^\dagger \nvDash \varphi$. Proposition 3.34(3) then implies that $\mathfrak{F} \nvDash \varphi^\dagger$. Thus, $\mathsf{MS4.t} \nvdash \varphi^\dagger$. □

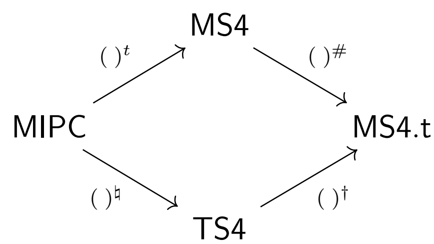
Remark 3.36.

1. The definition of the translation $(-)^\dagger : \mathsf{TS4} \to \mathsf{MS4.t}$ is suggested by the correspondence between TS4-frames and MS4.t-frames. Indeed, given an MS4.t-frame $\mathfrak{F}$, the relation Q_E in $\mathfrak{F}^\dagger$ is the composition of R and E, and the inverse relation $Q_E^\smile$ is the composition of E and $R^\smile$. Therefore, the modalities $\blacksquare_F$ and $\blacksquare_P$ are translated as $\Box_F\forall$ and $\forall\Box_P$, respectively.

2. It is natural to consider a modification of $(-)^\dagger$ where $\blacksquare_P$ is translated as $\Box_P\forall$. However, Theorem 3.35 fails for this modification. Nevertheless, its composition with $(-)^\natural$: $\mathsf{MIPC} \to \mathsf{TS4}$ is full and faithful, as we will see at the end of Section 3.4.3.

3.4.3 Translations of MIPC into MS4.t

We denote the composition of $(-)^\#$ and $(-)^t$ by $(-)^{t\#}$, and the composition of $(-)^\dagger$ and $(-)^\natural$ by $(-)^{\natural\dagger}$. Since we proved that all these four translations are full and faithful, we also have that $(-)^{t\#}$ and $(-)^{\natural\dagger}$ are full and faithful translations of MIPC into MS4.t. We have thus obtained the following diagram of full and faithful translations. We next show that this diagram is commutative up to logical equivalence in MS4.t.



Lemma 3.37. *For any formula φ of $\mathcal{L}_{\forall\exists}$, we have*

$$\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Diamond_P \varphi^{t\#}.$$

Proof. By Lemma 2.29 and Theorem 3.31, $\mathsf{MS4.t} \vdash \varphi^{t\#} \to \Box_F \varphi^{t\#}$. Therefore, $\mathsf{MS4.t} \vdash \Diamond_P \varphi^{t\#} \to \Diamond_P \Box_F \varphi^{t\#}$. The tense axiom then gives $\mathsf{MS4.t} \vdash \Diamond_P \varphi^{t\#} \to \varphi^{t\#}$. Thus, $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Diamond_P \varphi^{t\#}$. $\Box$

Theorem 3.38. *For any $\mathcal{L}_{\forall\exists}$-formula χ we have*

$$\mathsf{MS4.t} \vdash \chi^{t\#} \leftrightarrow \chi^{\natural\dagger}.$$

Proof. The two compositions compare as follows:

$$\bot^{t\#} = \bot \qquad\qquad \bot^{\natural\dagger} = \bot$$

$$p^{t\#} = \Box_F p \qquad\qquad p^{\natural\dagger} = \Box_F p$$

$$(\varphi \wedge \psi)^{t\#} = \varphi^{t\#} \wedge \psi^{t\#} \qquad\qquad (\varphi \wedge \psi)^{\natural\dagger} = \varphi^{\natural\dagger} \wedge \psi^{\natural\dagger}$$

$$(\varphi \vee \psi)^{t\#} = \varphi^{t\#} \vee \psi^{t\#} \qquad\qquad (\varphi \vee \psi)^{\natural\dagger} = \varphi^{\natural\dagger} \vee \psi^{\natural\dagger}$$

$$(\varphi \to \psi)^{t\#} = \Box_F(\neg\varphi^{t\#} \vee \psi^{t\#}) \qquad\qquad (\varphi \to \psi)^{\natural\dagger} = \Box_F(\neg\varphi^{\natural\dagger} \vee \psi^{\natural\dagger})$$

$$(\forall\varphi)^{t\#} = \Box_F \forall \varphi^{t\#} \qquad (\forall\varphi)^{\natural\dagger} = \Box_F \forall \varphi^{\natural\dagger}$$

$$(\exists\varphi)^{t\#} = \exists \varphi^{t\#} \qquad (\exists\varphi)^{\natural\dagger} = (\blacklozenge_P \varphi^{\natural})^{\dagger} = (\neg \blacksquare_P \neg \varphi^{\natural})^{\dagger}$$

$$= \neg \forall \Box_P \neg \varphi^{\natural\dagger}$$

Thus, they are identical except the $\exists$-clause. Therefore, to prove that $\mathsf{MS4.t} \vdash \chi^{t\#} \leftrightarrow \chi^{\natural\dagger}$ it is sufficient to prove that $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \varphi^{\natural\dagger}$ implies $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \neg\forall\Box_P\neg\varphi^{\natural\dagger}$. Since $\mathsf{MS4.t} \vdash \neg\forall\Box_P\neg\varphi^{\natural\dagger} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$, it is enough to prove that $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$. From the assumption $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \varphi^{\natural\dagger}$ it follows that $\mathsf{MS4.t} \vdash \exists\Diamond_P\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{\natural\dagger}$. By Lemma 3.37, $\mathsf{MS4.t} \vdash \varphi^{t\#} \leftrightarrow \Diamond_P\varphi^{t\#}$ and hence $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \exists\Diamond_P\varphi^{t\#}$. □

As we pointed out in Remark 3.36(2), there is another natural translation of MIPC into $\mathsf{MS4.t}$.

Definition 3.39. Let $(-)^{\flat} : \mathsf{MIPC} \to \mathsf{MS4.t}$ be the translation that differs from $(-)^{t\#}$ and $(-)^{\natural\dagger}$ only in the $\exists$-clause:

$$(\exists\varphi)^{\flat} = \Diamond_P \exists \varphi^{\flat}.$$

The translation $(-)^{\flat}$ provides a temporal interpretation of intuitionistic monadic quantifiers that is similar to the translation $(-)^{\natural}$ (see also Section 6).

Theorem 3.40. *For any $\mathcal{L}_{\forall\exists}$-formula χ we have*

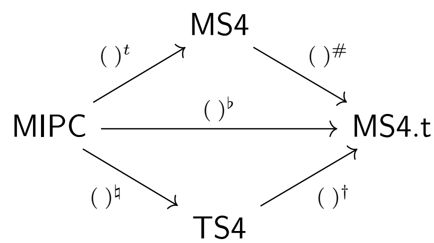
$$\mathsf{MS4.t} \vdash \chi^{\flat} \leftrightarrow \chi^{t\#}.$$

Consequently, the translation $(-)^{\flat}$ of MIPC into $\mathsf{MS4.t}$ is full and faithful.

Proof. The translations $()^{\flat}$ and $(-)^{t\#}$ are identical except the $\exists$-clause. Therefore, to prove that $\mathsf{MS4.t} \vdash \chi^{\flat} \leftrightarrow \chi^{t\#}$ it is sufficient to prove that $\mathsf{MS4.t} \vdash \varphi^{\flat} \leftrightarrow \varphi^{t\#}$ implies $\mathsf{MS4.t} \vdash$

$\Diamond_P \exists\varphi^\flat \leftrightarrow \exists\varphi^{t\#}$. By Lemma 3.37, $\mathsf{MS4.t} \vdash (\exists\varphi)^{t\#} \leftrightarrow \Diamond_P(\exists\varphi)^{t\#}$ which means $\mathsf{MS4.t} \vdash \exists\varphi^{t\#} \leftrightarrow \Diamond_P\exists\varphi^{t\#}$. From the assumption $\mathsf{MS4.t} \vdash \varphi^\flat \leftrightarrow \varphi^{t\#}$ it follows that $\mathsf{MS4.t} \vdash \Diamond_P\exists\varphi^\flat \leftrightarrow \Diamond_P\exists\varphi^{t\#}$. Thus, $\mathsf{MS4.t} \vdash \Diamond_P\exists\varphi^\flat \leftrightarrow \exists\varphi^{t\#}$. Since $(-)^{t\#}$ is full and faithful, it follows that $(-)^\flat$ is full and faithful as well. □

As a result, we obtain the following diagram of full and faithful translations that is commutative up to logical equivalence in MS4.t.



3.5 Finite model property

We are now ready to prove that the logics studied in Sections 2 and 3 all have the fmp. Our strategy is to first establish the fmp for MS4.t, and then use the full and faithful translations to conclude that all the logics we have considered have the fmp.

Let $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ be an MS4.t-algebra and $S \subseteq B$ a finite subset. Then $(B, \forall)$ is an S5-algebra. Let $(B', \forall')$ be the S5-subalgebra of $(B, \forall)$ generated by S. It is well known (see [8]) that $(B', \forall')$ is finite. Define $\Box'_F$ and $\Box'_P$ on B' by

$$\Box'_F a = \bigvee\{b \in B' \cap H_F \mid b \leq a\}$$

$$\Box'_P a = \bigvee\{b \in B' \cap H_P \mid b \leq a\}.$$

Definition 3.41. For an MS4.t-algebra $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ and $S \subseteq B$ a finite subset, let $\mathfrak{B}_S$ denote $(B', \Box'_F, \Box'_P, \forall')$.

Lemma 3.42. *$\mathfrak{B}_S$ is an* MS4.t-*algebra.*

Proof. By definition, $(B', \forall')$ is an $\mathsf{S5}$-algebra. Since $(B, \Box_F)$ and $(B, \Box_P)$ are both $\mathsf{S4}$-algebras, a standard argument (see [91, Lem. 4.14]) shows that $(B', \Box'_F)$ and $(B', \Box'_P)$ are also $\mathsf{S4}$-algebras. We show that $(B', \Box'_F, \Box'_P)$ is an $\mathsf{S4.t}$-algebra. Let H_F be the algebra of $\Box_F$-fixpoints and H_P the algebra of $\Box_P$-fixpoints of $\mathfrak{B}$. As noted in Remark 3.4, $\neg$ is a dual isomorphism between H_F and H_P. Therefore,

$$\begin{aligned}\Diamond'_F a := \neg\Box'_F\neg a &= \neg \bigvee\{b \in B' \cap H_F \mid b \le \neg a\}\\ &= \neg \bigvee\{b \in B' \cap H_F \mid a \le \neg b\}\\ &= \bigwedge\{\neg b \mid b \in B' \cap H_F,\ a \le \neg b\}\\ &= \bigwedge\{c \in B' \cap H_P \mid a \le c\}.\end{aligned}$$

Since this meet is finite and $\Box_P$ commutes with finite meets, we obtain

$$\begin{aligned}\Box_P\Diamond'_F a &= \Box_P\left(\bigwedge\{c \in B' \cap H_P \mid a \le c\}\right)\\ &= \bigwedge\{\Box_P c \mid c \in B' \cap H_P,\ a \le c\}\\ &= \bigwedge\{c \in B' \cap H_P \mid a \le c\}\\ &= \Diamond'_F a.\end{aligned}$$

Thus, $\Diamond'_F a \in B' \cap H_P$ which yields

$$\Box'_P\Diamond'_F a = \bigvee\{b \in B' \cap H_P \mid b \le \Diamond'_F a\} = \Diamond'_F a.$$

Similarly, we have that $\Diamond'_P a = \bigwedge\{c \in B' \cap H_F \mid a \le c\}$ from which we deduce that $\Box'_F\Diamond'_P a = \Diamond'_P a$. This implies that $a \le \Box'_P\Diamond'_F a$ and $a \le \Box'_F\Diamond'_P a$. Consequently, $(B, \Box'_F, \Box'_P)$ is an $\mathsf{S4.t}$-algebra.

It remains to show that $\Box'_F\forall' a \le \forall'\Box'_F a$ holds in $\mathfrak{B}_S$. For this it is sufficient to show that the set $B'_0 := B' \cap B_0$ of the $\forall'$-fixpoints of B' is an $\mathsf{S4}$-subalgebra of $(B', \Box'_F)$ because then

$\Box'_F\forall' a = \forall'\Box'_F\forall' a \leq \forall'\Box'_F a$. Suppose that $d \in B'_0$. Then $\Box'_F d = \bigvee\{b \in B' \cap H_F \mid b \leq d\}$. Let $b \in B' \cap H_F$. By Lemma 2.19(4), $\exists b = \exists\Box_F b = \Box_F\exists\Box_F b = \Box_F\exists b$. Therefore, $\exists b \in B' \cap H_F$. Moreover, $b \leq \exists b$ and $b \leq d$ implies $\exists b \leq \exists d = d$. Thus, $\Box'_F d = \bigvee\{\exists b \mid b \in B' \cap H_F,\ b \leq d\}$. Since $(B', \forall')$ is an $\mathsf{S5}$-algebra, B'_0 is the set of $\exists'$-fixpoints of B' and is closed under finite joins. Consequently, $\Box'_F d \in B'_0$ and so B'_0 is an $\mathsf{S4}$-subalgebra of $(B', \Box'_F)$. $\Box$

Theorem 3.43. $\mathsf{MS4.t}$ *has the fmp.*

Proof. It is sufficient to prove that each $\mathcal{L}_{T\forall}$-formula φ refuted on some $\mathsf{MS4.t}$-algebra is also refuted on a finite $\mathsf{MS4.t}$-algebra. Let $t(x_1, \ldots, x_n)$ be the term in the language of $\mathsf{MS4.t}$-algebras that corresponds to φ, and suppose there is an $\mathsf{MS4.t}$-algebra $\mathfrak{B} = (B, \Box_F, \Box_P, \forall)$ and $a_1, \ldots, a_n \in B$ such that $t(a_1, \ldots, a_n) \neq 1$ in $\mathfrak{B}$. Let

$$S = \{t'(a_1, \ldots, a_n) \mid t' \text{ is a subterm of } t\}.$$

Then S is a finite subset of B. Therefore, by Lemma 3.42, $\mathfrak{B}_S = (B', \Box'_F, \Box'_P, \forall)$ is a finite $\mathsf{MS4.t}$-algebra. It follows from the definition of $\Box'_F$ that, for each $b \in B'$, if $\Box_F b \in B'$, then $\Box'_F b = \Box_F b$. Similarly, if $\Box_P b \in B$, then $\Box'_P b = \Box_P b$. Thus, for each subterm t' of t, the computation of t' in $\mathfrak{B}_S$ is the same as that in $\mathfrak{B}$. Consequently, $t(a_1, \ldots, a_n) \neq 1$ in $\mathfrak{B}_S$, and we have found a finite $\mathsf{MS4.t}$-algebra refuting φ. $\Box$

Remark 3.44. Lemma 3.42 in particular proves that $\mathfrak{B}_S$ is an $\mathsf{S4.t}$-algebra. Thus, the proof of the fmp for $\mathsf{MS4.t}$ contains the proof of the fmp for $\mathsf{S4.t}$. In fact, $\mathsf{MS4.t}$ is a conservative extension of $\mathsf{S4.t}$.

We conclude this section by showing that the fmp for $\mathsf{TS4}$, $\mathsf{MS4}$, and MIPC is a consequence of Theorem 3.43.

Theorem 3.45.

1. *TS4 has the fmp.*

2. *MS4 has the fmp.*

3. *MIPC has the fmp.*

Proof. (1). Suppose that TS4 $\nvdash \varphi$. By Theorem 3.35, MS4.t $\nvdash \varphi^\dagger$. Since MS4.t has the fmp, there is a finite MS4.t-algebra $\mathfrak{B}$ such that $\mathfrak{B} \nvDash \varphi^\dagger$. As noted in Remark 3.29, $\mathfrak{B}$ is isomorphic to $(\mathfrak{B}_+)^+$. This yields that $\mathfrak{B}_+ \nvDash \varphi^\dagger$. By Proposition 3.34(2), $(\mathfrak{B}_+)^\dagger \nvDash \varphi$. We have thus obtained a finite TS4-frame $(\mathfrak{B}_+)^\dagger$ refuting φ. So $((\mathfrak{B}_+)^\dagger)^+$ is a finite TS4-algebra such that $((\mathfrak{B}_+)^\dagger)^+ \nvDash \varphi$.

(2). Similar to the proof of (1) but uses the translation $(-)^\#$: MS4 $\to$ MS4.t instead of $(-)^\dagger$.

(3). Similar to the proof of (1) but uses the composition $(-)^{t\#}$: MIPC $\to$ MS4.t instead of $(-)^\dagger$. Alternatively, we can use the other translations $(-)^{\natural\dagger}$ and $(-)^\flat$ of MIPC into MS4.t. $\square$

4 Temporal interpretation of predicate intuitionistic quantifiers

We now focus on the full predicate setting. It is well known that the predicate version of the Gödel translation is a full and faithful translation of the predicate intuitionistic logic IQC into the predicate modal logic $\mathsf{QS4}$. In this section we describe a translation of IQC into a tense predicate logic that realizes the temporal interpretation of the intuitionistic quantifiers as "always in the future" for $\forall$ and "sometime in the past" for $\exists$. In this setting additional care is needed in the choice of the tense predicate logic that will be the target of this translation. After a discussion about axiomatizations of predicate modal logics and their relational semantics, we define the tense predicate logic $\mathsf{Q}^{\circ}\mathsf{S4.t}$. We obtain a relational semantics for $\mathsf{Q}^{\circ}\mathsf{S4.t}$ by adapting the generalized semantics studied by Corsi [41]. Using a combination of syntactic and semantic methods, we show that the temporal translation is full and faithful on sentences. We also discuss how to connect the results of this section to those in Section 3. We end the first part of the thesis by mentioning some open problems and future directions of research related to this line of research.

4.1 IQC

Let $\mathcal{L}'$ be the language consisting of countably many individual variables $x, y, \ldots$, countably many n-ary predicate symbols $P, Q, \ldots$ (for each $n \geq 0$), the logical connectives $\bot, \wedge, \vee, \to$, and the quantifiers $\forall, \exists$.

Formulas are defined as usual by induction and are denoted with upper case letters $A, B, \ldots$. Let x, y be individual variables and A a formula. If x is a free variable of A and does not occur in the scope of $\forall y$ or $\exists y$, then we denote by $A(y/x)$ the formula obtained from A by replacing all the free occurrences of x by y.

The following definition of the intuitionistic predicate logic IQC is taken from [60, Sec 2.6]. We point out that, unlike [60], we prefer to work with axiom schemes, and hence do not need the inference rule of substitution.

Definition 4.1. The intuitionistic predicate logic IQC is the least set of formulas of $\mathcal{L}'$ containing all substitution instances of theorems of IPC, the axiom schemes

1. $\forall x A \to A(y/x)$ Universal instantiation (UI)
2. $A(y/x) \to \exists x A$
3. $\forall x(A \to B) \to (A \to \forall x B)$ with x not free in A
4. $\forall x(A \to B) \to (\exists x A \to B)$ with x not free in B

and closed under the inference rules of Modus Ponens (MP) and

$$\frac{A}{\forall x A} \qquad \text{Generalization (Gen)}$$

We next describe Kripke semantics for IQC (see [83, 56]).

Definition 4.2. An IQC-*frame* is a triple $\mathfrak{F} = (W, R, D)$ where

- W is a nonempty set whose elements are called the *worlds* of $\mathfrak{F}$.
- R is a partial order on W.
- D is a function that associates to each $w \in W$ a nonempty set D_w such that wRv implies $D_w \subseteq D_v$ for each $w, v \in W$. The set D_w is called the *domain* of w.

Definition 4.3.

- An *interpretation* of $\mathcal{L}'$ in $\mathfrak{F}$ is a function I associating to each world w and n-ary predicate symbol P an n-ary relation $I_w(P) \subseteq (D_w)^n$ such that wRv implies $I_w(P) \subseteq I_v(P)$.

- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where $\mathfrak{F}$ is an IQC-frame and I is an interpretation in $\mathfrak{F}$.

- Let w be a world of $\mathfrak{F}$. A *w-assignment* is a function σ associating to each individual variable x an element $\sigma(x)$ of D_w. Note that if wRv, then σ is also a v-assignment.

- Let σ and τ be two w-assignments and x an individual variable. Then τ is said to be an *x-variant* of σ if $\tau(y) = \sigma(y)$ for all $y \neq x$.

We next recall the definition of when a formula A is true in a world w of a model $\mathfrak{M} = (\mathfrak{F}, I)$ under the w-assignment σ, written $\mathfrak{M} \models_w^\sigma A$.

Definition 4.4.

$\mathfrak{M} \models_w^\sigma \bot$		never
$\mathfrak{M} \models_w^\sigma P(x_1, \ldots, x_n)$	iff	$(\sigma(x_1), \ldots, \sigma(x_n)) \in I_w(P)$
$\mathfrak{M} \models_w^\sigma B \wedge C$	iff	$\mathfrak{M} \models_w^\sigma B$ and $\mathfrak{M} \models_w^\sigma C$
$\mathfrak{M} \models_w^\sigma B \vee C$	iff	$\mathfrak{M} \models_w^\sigma B$ or $\mathfrak{M} \models_w^\sigma C$
$\mathfrak{M} \models_w^\sigma B \to C$	iff	for all v with wRv, if $\mathfrak{M} \models_v^\sigma B$, then $\mathfrak{M} \models_v^\sigma C$
$\mathfrak{M} \models_w^\sigma \forall x B$	iff	for all v with wRv and each v-assignment τ that is an x-variant of σ, $\mathfrak{M} \models_v^\tau B$
$\mathfrak{M} \models_w^\sigma \exists x B$	iff	there exists a w-assignment τ that is an x-variant of σ such that $\mathfrak{M} \models_w^\tau B$

Definition 4.5.

- We say that A is *true* in a world w of $\mathfrak{M}$, written $\mathfrak{M} \vDash_w A$, if for all w-assignments σ, we have $\mathfrak{M} \vDash_w^\sigma A$.
- We say that A is *true* in $\mathfrak{M}$, written $\mathfrak{M} \vDash A$, if for all worlds $w \in W$, we have $\mathfrak{M} \vDash_w A$.
- We say that A is *valid* in a frame $\mathfrak{F}$, written $\mathfrak{F} \vDash A$, if for all models $\mathfrak{M}$ based on $\mathfrak{F}$, we have $\mathfrak{M} \vDash A$.

We have the following well-known completeness of IQC with respect to Kripke semantics.

Theorem 4.6 ([83]). *The intuitionistic predicate logic* IQC *is sound and complete with respect to Kripke semantics; that is, for each formula A,*

$$\mathsf{IQC} \vdash A \text{ iff } \mathfrak{F} \vDash A \text{ for each } \mathsf{IQC}\text{-frame } \mathfrak{F}.$$

4.2 Modal predicate logics

Modal predicate logics were first studied by Barcan [7] and Carnap [39] in 1940s. Algebraic and topological semantics of modal predicate logics were studied by Rasiowa and Sikorski (see [100]). Relational semantics of modal predicate logics was initiated by Kripke [81, 82] in late 1950s/early 1960s. In 1959 Kripke [81] proved Kripke completeness of predicate $\mathsf{S5}$. In late 1960s Cresswell [42, 43] (see also Hughes and Cresswell [74]), Schütte [102], and Thomason [109] proved Kripke completeness of predicate T and $\mathsf{S4}$. Gabbay [55] proved Kripke completeness of some predicate modal logics with respect to frames with constant domains. Since then many completeness results have been obtained with respect to Kripke semantics, but there is also a large body of incompleteness results, which is one of the reasons that the model theory of modal predicate logic is less advanced than that of modal propositional logic (see, e.g., [60, 61] and the references therein).

Let K be the least normal modal propositional logic and let QK be the standard predicate extension of K. The language $\mathcal{L}'_\Box$ of QK is the extension of $\mathcal{L}'$ with the modality $\Box$. Since the modal logics we consider are based on the classical logic, it is sufficient to only consider the logical connectives $\bot, \to$ and the quantifier $\forall$. The logical connectives $\wedge, \vee, \neg, \leftrightarrow$, the quantifier $\exists$, and the modality $\Diamond$ are treated as usual abbreviations.

We next recall the definition of QK (see, e.g., [60, Sec 2.6], but note, as in Section 4.1, that we work with axiom schemes instead of having the inference rule of substitution).

Definition 4.7. The modal predicate logic QK is the least set of formulas of $\mathcal{L}'_\Box$ containing all substitution instances of theorems of K, the axiom schemes (i) and (iii) of Definition 4.1, and closed under (MP), (Gen), and (N).

The definition of QK-*frames* $\mathfrak{F} = (W, R, D)$ is the same as that of IQC-frames (see Definition 4.2) with the only difference that R can be an arbitrary relation. *Models* are also defined the same way, but without the requirement that wRv implies $I_w(P) \subseteq I_v(P)$. The connectives and quantifiers are interpreted at each world in the usual classical way, and

$$\mathfrak{M} \models^\sigma_w \Box A \text{ iff } (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \models^\sigma_v A).$$

Truth and *validity* of formulas are defined as usual.

Kripke completeness of QK was first established by Gabbay [55, Thm. 8.5]:

Theorem 4.8. *The modal predicate logic* QK *is sound and complete with respect to Kripke semantics.*

The modal predicate logic QS4 is defined by adding the S4 axioms to QK. That QS4 is sound and complete with respect to the class of QK-frames with a reflexive and transitive accessibility relation is known since the late 1960s, see [43, 102].

The Gödel translation extends to the predicate setting as follows.

$$\begin{array}{rcll} \bot^t & = & \bot & \\ P(x_1,\ldots,x_n)^t & = & \Box P(x_1,\ldots,x_n) & \quad \text{for each } n\text{-ary predicate symbol } P \\ (A\wedge B)^t & = & A^t \wedge B^t & \\ (A\vee B)^t & = & A^t \vee B^t & \\ (A\to B)^t & = & \Box(A^t \to B^t) & \\ (\forall x A)^t & = & \Box\forall x A^t & \\ (\exists x A)^t & = & \exists x A^t & \end{array}$$

The first proof of the faithfulness and fullness of the predicate Gödel translation is due to Rasiowa and Sikorski [99] (see also [100, XI.11.5]). Schütte [102] proved it using the relational semantics; see also [60, Sec. 2.11].

Theorem 4.9. *The Gödel translation of* IQC *into* QS4 *is full and faithful; that is,*

$$\mathsf{IQC} \vdash A \quad \textit{iff} \quad \mathsf{QS4} \vdash A^t.$$

4.3 Q°K

The following two principles play an important role in the study of modal predicate logics. They were first considered by Barcan [7].

$\forall x \Box A \to \Box\forall x A$	Barcan formula	(BF)
$\Box\forall x A \to \forall x \Box A$	converse Barcan formula	(CBF)

It is easy to see that CBF is a theorem of QK. Indeed, this follows from Theorem 4.8 and the fact that CBF is valid in each QK-frame because the domains of QK-frames are increasing. On the other hand, a QK-frame validates BF iff it has *constant domains*, meaning that wRv implies $D_w = D_v$, and we have the following well-known theorem (see, e.g., [55, Thm. 9.3]):

Theorem 4.10. *The logic* $\mathsf{QK} + \mathsf{BF}$ *is sound and complete with respect to the class of* QK-*frames with constant domains.*

A modal predicate logic whose Kripke frames have neither increasing nor decreasing domains was considered already by Kripke [82]. Building on this work, Hughes and Cresswell [73, pp. 304–309] introduced a similar predicate modal logic and proved its completeness with respect to a generalized Kripke semantics. Fitting and Mendelsohn [54, Sec. 6.2] gave an alternate axiomatization of this logic. Building on the work of Fitting and Mendelsohn, Corsi [41] defined the system $\mathsf{Q}^\circ\mathsf{K}$.

Definition 4.11. The logic $\mathsf{Q}^\circ\mathsf{K}$ is the least set of formulas of $\mathcal{L}'_\Box$ containing all substitution instances of theorems of K, the axiom schemes

1. $\forall y(\forall x A \to A(y/x))$ (UI°)

2. $\forall x(A \to B) \to (\forall x A \to \forall x B)$

3. $\forall x \forall y A \leftrightarrow \forall y \forall x A$

4. $A \to \forall x A$ with x not free in A

and closed under (MP), (Gen), and (N).

Remark 4.12. In Definition 4.11, replacing UI° with UI yields an equivalent definition of QK (see [41, p. 1487]). Therefore, $\mathsf{Q}^\circ\mathsf{K}$ is contained in QK.

Kripke frames for $\mathsf{Q}^\circ\mathsf{K}$ generalize Kripke frames for QK by having two domains, inner and outer.

Definition 4.13. A $\mathsf{Q}^\circ\mathsf{K}$-*frame* is a quadruple $\mathfrak{F} = (W, R, D, U)$ where

- (W, R) is a K-frame.

- D is a function that associates to each $w \in W$ a set D_w. The set D_w is called the *inner domain* of w.

- U is a nonempty set containing the union of all the D_w. The set U is called the *outer domain* of $\mathfrak{F}$.

Definition 4.13 is a particular case of the frames considered by Corsi [41] where increasing outer domains are allowed. For our purposes, taking a fixed outer domain U is sufficient. We recall from [41] how to interpret $\mathcal{L}'_\Box$ in a $\mathsf{Q}^\circ\mathsf{K}$-frame $\mathfrak{F} = (W, R, D, U)$.

Definition 4.14.

- An *interpretation* of $\mathcal{L}'_\Box$ in $\mathfrak{F}$ is a function I associating to each world w and an n-ary predicate symbol P an n-ary relation $I_w(P) \subseteq U^n$.

- A *model* is a pair $\mathfrak{M} = (\mathfrak{F}, I)$ where $\mathfrak{F}$ is a $\mathsf{Q}^\circ\mathsf{K}$-frame and I is an interpretation in $\mathfrak{F}$.

- An *assignment* in $\mathfrak{F}$ is a function σ that associates to each individual variable an element of U.

- If σ and τ are two assignments and x is an individual variable, τ is said to be an *x-variant* of σ if $\tau(y) = \sigma(y)$ for all $y \neq x$.

- We say that an assignment σ is *w-inner* for $w \in W$ if $\sigma(x) \in D_w$ for each individual variable x.

We next recall from [41] the definition of when a formula A is true in a world w of a model $\mathfrak{M} = (\mathfrak{F}, I)$ under the assignment σ, written $\mathfrak{M} \vDash^\sigma_w A$.

Definition 4.15.

$\mathfrak{M} \vDash_w^\sigma \bot$		never
$\mathfrak{M} \vDash_w^\sigma P(x_1, \dots, x_n)$	iff	$(\sigma(x_1), \dots, \sigma(x_n)) \in I_w(P)$
$\mathfrak{M} \vDash_w^\sigma B \to C$	iff	$\mathfrak{M} \vDash_w^\sigma B$ implies $\mathfrak{M} \vDash_w^\sigma C$
$\mathfrak{M} \vDash_w^\sigma \forall x B$	iff	for all x-variants τ of σ with $\tau(x) \in D_w$, $\mathfrak{M} \vDash_w^\tau B$
$\mathfrak{M} \vDash_w^\sigma \Box B$	iff	for all v such that wRv, $\mathfrak{M} \vDash_v^\sigma B$

Definition 4.16. A formula A is *true* in a model $\mathfrak{M} = (\mathfrak{F}, I)$ at the world $w \in W$ (in symbols $\mathfrak{M} \vDash_w A$) if for all assignments σ, we have $\mathfrak{M} \vDash_w^\sigma A$. The definition of *truth* in a model and *validity* in a frame are the same as in Definition 4.5.

We have the following completeness result for $\mathsf{Q}^\circ\mathsf{K}$, see [41, Thm. 1.32].

Theorem 4.17. $\mathsf{Q}^\circ\mathsf{K}$ *is sound and complete with respect to the class of* $\mathsf{Q}^\circ\mathsf{K}$*-frames.*

Definition 4.18. Let $\mathfrak{F} = (W, R, D, U)$ be a $\mathsf{Q}^\circ\mathsf{K}$-frame.

- We say that $\mathfrak{F}$ has *increasing inner domains* if wRv implies $D_w \subseteq D_v$ for each $w, v \in W$.
- We say that $\mathfrak{F}$ has *decreasing inner domains* if wRv implies $D_v \subseteq D_w$ for each $w, v \in W$.
- If $\mathfrak{F}$ has both increasing and decreasing inner domains, we say that $\mathfrak{F}$ has *constant inner domains.*

The following axiom scheme guarantees nonempty inner domains (hence the abbreviation):

$$\forall x A \to A \ \text{ with } x \text{ not free in } A \qquad (\mathsf{NID})$$

The next proposition is not difficult to verify (see, e.g., [54, Sec. 4.9] and [41, pp. 1487–1488]).

Proposition 4.19. *Let* $\mathfrak{F} = (W, R, D, U)$ *be a* $\mathsf{Q}^\circ\mathsf{K}$*-frame.*

- $\mathfrak{F}$ *validates* CBF *iff* $\mathfrak{F}$ *has increasing inner domains.*
- $\mathfrak{F}$ *validates* BF *iff* $\mathfrak{F}$ *has decreasing inner domains.*
- $\mathfrak{F}$ *validates* NID *iff* $\mathfrak{F}$ *has nonempty inner domains.*

We have the following completeness results for logics obtained by adding CBF, BF, and NID to $\mathsf{Q}^\circ\mathsf{K}$ (see [41, Thms. 1.30, 1.32, and Footnote 7]):

Theorem 4.20.

- $\mathsf{Q}^\circ\mathsf{K}+\mathsf{CBF}$ *is sound and complete with respect to the class of* $\mathsf{Q}^\circ\mathsf{K}$*-frames with increasing inner domains.*
- $\mathsf{Q}^\circ\mathsf{K} + \mathsf{CBF} + \mathsf{BF}$ *is sound and complete with respect to the class of* $\mathsf{Q}^\circ\mathsf{K}$*-frames with constant inner domains.*
- *Adding* NID *to the above two logics or to* $\mathsf{Q}^\circ\mathsf{K}$ *yields completeness of the resulting logics with respect to the corresponding classes of frames which have nonempty inner domains.*

On the other hand, completeness of $\mathsf{Q}^\circ\mathsf{K} + \mathsf{BF}$ remains open (see [41, p. 1510]).

4.4 $\mathsf{Q}^\circ\mathsf{S4.t}$

The tense predicate logic we will translate IQC into is based on the tense propositional logic $\mathsf{S4.t}$ discussed in Section 3.1. We use the temporal modalities $\Box_F$ ("always in the future") and $\Box_P$ ("always in the past"). $\Diamond_F$ ("sometime in the future") and $\Diamond_P$ ("sometime in the past") are the usual abbreviations of $\neg\Box_F\neg$ and $\neg\Box_P\neg$.

Let $\mathcal{L}'_T$ be the bimodal predicate language obtained by extending $\mathcal{L}'$ with the two modalities $\Box_F$ and $\Box_P$.

Definition 4.21. The logic $\mathsf{QS4.t}$ is the least set of formulas of $\mathcal{L}'_T$ containing all substitution instances of theorems of $\mathsf{S4.t}$, the axiom schemes (i) and (iii) of Definition 4.1, and closed under (MP), (Gen), ($\mathrm{N_F}$), and ($\mathrm{N_P}$).

The following are temporal versions of CBF and BF:

$\forall x \Box_F A \to \Box_F \forall x A$	Barcan formula for $\Box_F$	($\mathsf{BF_F}$)
$\Box_F \forall x A \to \forall x \Box_F A$	converse Barcan formula for $\Box_F$	($\mathsf{CBF_F}$)
$\forall x \Box_P A \to \Box_P \forall x A$	Barcan formula for $\Box_P$	($\mathsf{BF_P}$)
$\Box_P \forall x A \to \forall x \Box_P A$	converse Barcan formula for $\Box_P$	($\mathsf{CBF_P}$)

The proof that $\mathsf{QK} \vdash \mathsf{CBF}$ (see, e.g., [82, p. 88]) can be adapted to prove that $\mathsf{QS4.t} \vdash \mathsf{CBF_F}$ and $\mathsf{QS4.t} \vdash \mathsf{CBF_P}$. It is also known that $\mathsf{CBF_F}$ and $\mathsf{BF_P}$, as well as $\mathsf{CBF_P}$ and $\mathsf{BF_F}$ are derivable from each other in any tense predicate logic. Therefore, all four are theorems of $\mathsf{QS4.t}$. This is reflected in the fact that $\mathsf{QS4.t}$-frames have constant domains. Indeed, $\mathsf{QS4.t}$ is complete with respect to this semantics (see Section 4.7). But this is problematic for translating IQC fully into $\mathsf{QS4.t}$ since IQC-frames with constant domains validate the additional axiom $\forall x(A \vee B) \to (A \vee \forall x B)$, where x is not free in A, which is not a theorem of IQC (see, e.g., [56, p. 53, Cor. 8]).

Consequently, we need to work with a weaker logic than $\mathsf{QS4.t}$. To this end, we introduce the logic $\mathsf{Q^\circ S4.t}$, which weakens $\mathsf{QS4.t}$ the same way $\mathsf{Q^\circ K}$ weakens QK.

Definition 4.22. The logic $\mathsf{Q^\circ S4.t}$ is the least set of formulas of $\mathcal{L}'_T$ containing all substitution instances of theorems of $\mathsf{S4.t}$, the axiom schemes (i), (ii), (iii), (iv) of $\mathsf{Q^\circ K}$ (see Definition 4.11), NID, $\mathsf{CBF_F}$, and closed under (MP), (Gen), ($\mathrm{N_F}$), and ($\mathrm{N_P}$).

Proposition 4.23. $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \mathsf{BF_P}$.

Proof. We first show that $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \Diamond_F \forall x B \to \forall x \Diamond_F B$ for any formula B. We consider the proof

1. $\forall x(\forall x B \to B)$
2. $\forall x \Box_F(\forall x B \to B)$
3. $\Box_F(\forall x B \to B) \to (\Diamond_F \forall x B \to \Diamond_F B)$
4. $\forall x \Box_F(\forall x B \to B) \to \forall x(\Diamond_F \forall x B \to \Diamond_F B)$
5. $\forall x(\Diamond_F \forall x B \to \Diamond_F B)$
6. $\forall x \Diamond_F \forall x B \to \forall x \Diamond_F B$
7. $\Diamond_F \forall x B \to \forall x \Diamond_F B$

where 1 is an instance of UI°; 2 is obtained from 1 by adding $\Box_F$ inside $\forall x$ by applying ($\mathrm{N_F}$), $\mathsf{CBF_F}$, and (MP); 3 is a substitution instance of the K-theorem $\Box_F(C \to D) \to (\Diamond_F C \to \Diamond_F D)$ for $\Box_F$; 4 is obtained from 3 by first adding and then distributing $\forall x$ inside the implication by applying (Gen), axiom (ii) of $\mathsf{Q}^\circ\mathsf{K}$, and (MP); 5 follows from 2 and 4 by (MP); 6 is obtained from 5 by distributing $\forall x$; and 7 follows from 6 and axiom (iv) of $\mathsf{Q}^\circ\mathsf{K}$.

We now prove that $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall x \Box_P A \to \Box_P \forall x A$.

1. $\forall x \Box_P A \to \Box_P \Diamond_F \forall x \Box_P A$
2. $\Diamond_F \forall x \Box_P A \to \forall x \Diamond_F \Box_P A$
3. $\Box_P \Diamond_F \forall x \Box_P A \to \Box_P \forall x \Diamond_F \Box_P A$
4. $\Diamond_F \Box_P A \to A$
5. $\forall x \Diamond_F \Box_P A \to \forall x A$
6. $\Box_P \forall x \Diamond_F \Box_P A \to \Box_P \forall x A$
7. $\forall x \Box_P A \to \Box_P \forall x A$

where 1 is an instance of axiom (i) of $\mathsf{S4.t}$; 2 is an instance of $\Diamond_F \forall x B \to \forall x \Diamond_F B$ proved above; 3 and 6 follow from 2 and 5 by adding and distributing $\Box_P$ in the implication; 4 is an instance of the $\mathsf{S4.t}$-theorem $\Diamond_F \Box_P C \to C$; 5 is obtained from 4 by adding and distributing $\forall x$; and 7 follows from 1, 3, and 6. $\Box$

Definition 4.24. A $\mathsf{Q^\circ S4.t}$-*frame* is a $\mathsf{Q^\circ K}$-frame $\mathfrak{F} = (W, R, D, U)$ (see Definition 4.13) with nonempty increasing inner domains whose accessibility relation is reflexive and transitive.

Models and assignments are defined as in Definition 4.14. The clauses of when a formula A of $\mathcal{L}'_T$ is true in a world w of a $\mathsf{Q^\circ S4.t}$-model $\mathfrak{M} = (\mathfrak{F}, I)$ under the assignment σ, written $\mathfrak{M} \vDash^\sigma_w A$, are defined as in Definition 4.15, but we replace the $\Box$-clause with the following two clauses:

$$\mathfrak{M} \vDash^\sigma_w \Box_F B \quad \text{iff} \quad (\forall v \in W)(wRv \Rightarrow \mathfrak{M} \vDash^\sigma_v B)$$
$$\mathfrak{M} \vDash^\sigma_w \Box_P B \quad \text{iff} \quad (\forall v \in W)(vRw \Rightarrow \mathfrak{M} \vDash^\sigma_v B)$$

For formulas of $\mathcal{L}'_T$ we define truth in a model and validity in a frame as in Definition 4.16.

Theorem 4.25. $\mathsf{Q^\circ S4.t}$ *is sound with respect to the class of* $\mathsf{Q^\circ S4.t}$-*frames; that is, for each formula A of $\mathcal{L}'_T$ and* $\mathsf{Q^\circ S4.t}$-*frame $\mathfrak{F}$, from* $\mathsf{Q^\circ S4.t} \vdash A$ *it follows that* $\mathfrak{F} \vDash A$.

Proof. It is sufficient to show that each axiom scheme is valid in all $\mathsf{Q^\circ S4.t}$-frames and that each rule of inference preserves validity. This can be done by direct verification. We only show that the axiom scheme $\mathsf{CBF_F}$ is valid in all $\mathsf{Q^\circ S4.t}$-frames. Let $\mathfrak{M} = (\mathfrak{F}, I)$ be a $\mathsf{Q^\circ S4.t}$-model, $w \in W$, and σ an assignment. If $\mathfrak{M} \vDash^\sigma_w \Box_F \forall x A$, then for all v with wRv we have $\mathfrak{M} \vDash^\sigma_v \forall x A$. This implies that for each x-variant τ of σ with $\tau(x) \in D_v$ we have $\mathfrak{M} \vDash^\tau_v A$. Since $D_w \subseteq D_v$, this is in particular true for x-variants τ of σ with $\tau(x) \in D_w$. Therefore, for each x-variant τ of σ with $\tau(x) \in D_w$ and for each v with wRv we have $\mathfrak{M} \vDash^\tau_v A$. Thus, for each x-variant τ of σ with $\tau(x) \in D_w$, we have $\mathfrak{M} \vDash_w^\sigma \Box_F A$. Consequently, $\mathfrak{M} \vDash_w^\sigma \forall x \Box_F A$. This shows that $\mathfrak{F} \vDash \Box_F \forall x A \to \forall x \Box_F A$ for each $\mathsf{Q^\circ S4.t}$-frame $\mathfrak{F}$. □

On the other hand, completeness of $\mathsf{Q^\circ S4.t}$ remains an interesting open problem, which is related to the open problem of completeness of $\mathsf{Q^\circ K} + \mathsf{BF}$ (see Section 4.7).

4.5 The temporal translation of IQC into $\mathsf{Q^\circ S4.t}$

In this section we prove our main result that the translation obtained by modifying the Gödel translation on the quantifiers as follows

$$\begin{array}{rcl} (\forall x A)^t & = & \Box_F \forall x A^t \\ (\exists x A)^t & = & \Diamond_P \exists x A^t \end{array}$$

translates IQC into $\mathsf{Q^\circ S4.t}$ fully and faithfully. Our strategy is to prove faithfulness of the translation syntactically, while fullness will be proved by semantical means, utilizing Kripke completeness of IQC.

Our syntactic proof of faithfulness is based on a series of technical lemmas. To keep the notation simple, we denote lists of variables by bold letters. If $\mathbf{x} = x_1, \ldots, x_n$, we write $\forall \mathbf{x}$ for $\forall x_1 \cdots \forall x_n$. We point out that it is a consequence of axioms (ii) and (iii) of $\mathsf{Q^\circ K}$ that from the point of view of provability in $\mathsf{Q^\circ S4.t}$, the order of variables in $\forall \mathbf{x}$ does not matter.

Lemma 4.26. *If A is a formula of $\mathcal{L}'$, then $\mathsf{Q^\circ S4.t} \vdash A^t \to \Box_F A^t$ and $\mathsf{Q^\circ S4.t} \vdash \Diamond_P A^t \to A^t$.*

Proof. We only prove that $\mathsf{Q^\circ S4.t} \vdash A^t \to \Box_F A^t$ since it implies that $\mathsf{Q^\circ S4.t} \vdash \Diamond_P A^t \to A^t$. The proof is by induction on the complexity of A. If $A = \bot$, then $A^t = \bot$ and it is clear that $\mathsf{Q^\circ S4.t} \vdash \bot \to \Box_F \bot$.

If A is either an atomic formula $P(x_1, \ldots, x_n)$ or of the form $B \to C$ or $\forall x B$, then A^t is of the form $\Box_F D$. Therefore, the 4-axiom $\Box_F D \to \Box_F \Box_F D$ implies that in all these cases $\mathsf{Q}^\circ\mathsf{S4.t} \vdash A^t \to \Box_F A^t$.

If $A = \exists x B$, then $A^t = \Diamond_P \exists x B^t$. So $\Box_F A^t = \Box_F \Diamond_P \exists x B^t$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \Diamond_P \exists x B^t \to \Box_F \Diamond_P \exists x B^t$ because it is a substitution instance of the $\mathsf{S4.t}$-theorem $\Diamond_P C \to \Box_F \Diamond_P C$. Finally, if $A = B \wedge C$ or $A = B \vee C$, then we have $A^t = B^t \wedge C^t$ or $A^t = B^t \vee C^t$. By inductive hypothesis, $\mathsf{Q}^\circ\mathsf{S4.t} \vdash B^t \to \Box_F B^t$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash C^t \to \Box_F C^t$. Since $\mathsf{Q}^\circ\mathsf{S4.t} \vdash (\Box_F B^t \wedge \Box_F C^t) \to \Box_F (B^t \wedge C^t)$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash (\Box_F B^t \vee \Box_F C^t) \to \Box_F (B^t \vee C^t)$, we obtain $\mathsf{Q}^\circ\mathsf{S4.t} \vdash (B^t \wedge C^t) \to \Box_F (B^t \wedge C^t)$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash (B^t \vee C^t) \to \Box_F (B^t \vee C^t)$. □

Lemma 4.27. *The following are theorems of* $\mathsf{Q}^\circ\mathsf{S4.t}$*:*

1. $\forall y (A(y/x) \to \exists x A)$.

2. $\forall x (A \to B) \to (A \to \forall x B)$ *if x is not free in A.*

3. $\forall x (A \to B) \to (\exists x A \to B)$ *if x is not free in B.*

Proof. Follows from [41, Lem. 1.3]. □

Lemma 4.28. *For formulas A, B of $\mathcal{L}'$, the following are theorems of* $\mathsf{Q}^\circ\mathsf{S4.t}$.

1. $\Box_F (\Box_F \forall x A^t \to A^t)$ *if x is not free in A.*

2. $\forall y \Box_F (\Box_F \forall x A^t \to A(y/x)^t)$.

3. $\Box_F (A^t \to \Diamond_P \exists x A^t)$ *if x is not free in A.*

4. $\forall y \Box_F (A(y/x)^t \to \Diamond_P \exists x A^t)$.

5. $\Box_F(\Box_F\forall x\Box_F(A^t \to B^t) \to \Box_F(A^t \to \Box_F\forall x B^t))$ *if x is not free in A.*

6. $\Box_F(\Box_F\forall x\Box_F(A^t \to B^t) \to \Box_F(\Diamond_P\exists x A^t \to B^t))$ *if x is not free in B.*

Proof. Note that x is free in A iff it is free in A^t, and $A(y/x)^t = A^t(y/x)$.

(i). We have the proof

1. $\forall x A^t \to A^t$

2. $\Box_F\forall x A^t \to A^t$

3. $\Box_F(\Box_F\forall x A^t \to A^t)$

where 1 is an instance of NID because x is not free in A^t; 2 is obtained from 1 by applying the T-axiom for $\Box_F$; 3 is obtained from 2 by ($\mathrm{N_F}$).

(ii). We have the proof

1. $\forall y(\forall x A^t \to A^t(y/x))$

2. $\forall y(\Box_F\forall x A^t \to A^t(y/x))$

3. $\forall y\Box_F(\Box_F\forall x A^t \to A^t(y/x))$

where 1 is an instance of UI°; 2 follows from 1 by applying the T-axiom for $\Box_F$ inside $\forall y$; 3 is obtained from 2 by introducing $\Box_F$ inside $\forall y$.

(iii). We have the proof

1. $A^t \to \exists x A^t$

2. $A^t \to \Diamond_P\exists x A^t$

3. $\Box_F(A^t \to \Diamond_P\exists x A^t)$

where 1 is an instance of $C \to \exists x C$, with x not free in C, which is equivalent to NID; 2 follows from 1 by the T-axiom for $\Diamond_P$; 3 is obtained from 2 by ($\mathrm{N_F}$).

(iv). We have the proof

1. $\forall y(A^t(y/x) \to \exists x A^t)$

2. $\forall y(A^t(y/x) \to \Diamond_P \exists x A^t)$

3. $\forall y \Box_F (A^t(y/x) \to \Diamond_P \exists x A^t)$

where 1 follows from Lemma 4.27(i); 2 follows from 1 by applying the T-axiom for $\Diamond_P$ inside $\forall y$; 3 is obtained from 2 by introducing $\Box_F$ inside $\forall y$.

(v). We have the proof

1. $\forall x(A^t \to B^t) \to (A^t \to \forall x B^t)$

2. $\forall x \Box_F (A^t \to B^t) \to (A^t \to \forall x B^t)$

3. $\Box_F \forall x \Box_F (A^t \to B^t) \to (\Box_F A^t \to \Box_F \forall x B^t)$

4. $\Box_F \forall x \Box_F (A^t \to B^t) \to (A^t \to \Box_F \forall x B^t)$

5. $\Box_F \forall x \Box_F (A^t \to B^t) \to \Box_F (A^t \to \Box_F \forall x B^t)$

6. $\Box_F (\Box_F \forall x \Box_F (A^t \to B^t) \to \Box_F (A^t \to \Box_F \forall x B^t))$

where 1 follows from Lemma 4.27(ii); 2 follows from 1 by applying the T-axiom for $\Box_F$; 3 is obtained from 2 by adding and distributing $\Box_F$; 4 follows from 3 by Lemma 4.26; 5 is obtained from 4 by adding and distributing $\Box_F$ and getting rid of one $\Box_F$ in the antecedent using the 4-axiom; 6 follows from 5 by ($\mathrm{N_F}$).

(vi). We have the proof

1. $\forall x(A^t \to B^t) \to (\exists x A^t \to B^t)$

2. $\forall x(A^t \to B^t) \to (\exists x \Diamond_P A^t \to B^t)$

3. $\forall x \Box_F (A^t \to B^t) \to (\exists x \Diamond_P A^t \to B^t)$

4. $\forall x \Box_F (A^t \to B^t) \to (\Diamond_P \exists x A^t \to B^t)$

5. $\Box_F \forall x \Box_F (A^t \to B^t) \to \Box_F (\Diamond_P \exists x A^t \to B^t)$

6. $\Box_F (\Box_F \forall x \Box_F (A^t \to B^t) \to \Box_F (\Diamond_P \exists x A^t \to B^t))$

where 1 follows from Lemma 4.27(iii); 2 follows from 1 by Lemma 4.26; 3 follows from 2 by applying the T-axiom for $\Box_F$; 4 follows from 3 and the fact that $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \Diamond_P \exists x A^t \to \exists x \Diamond_P A^t$ because it is a consequence of $\mathsf{BF_P}$; 5 is obtained from 4 by adding and distributing $\Box_F$; 6 follows from 5 by ($\mathrm{N_F}$). ☐

Lemma 4.29. *If C is an instance of an axiom scheme of* IQC *and* $\mathbf{x}$ *is the list of free variables in C, then* $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{x}\, C^t$.

Proof. If C is an instance of a theorem of IPC, then it follows from the faithfulness of the Gödel translation in the propositional case that C^t is a theorem of $\mathsf{Q}^\circ\mathsf{S4.t}$ (since $\Box_F$ is an $\mathsf{S4}$-modality). Applying (Gen) to each free variable of C^t then yields a proof of $\forall \mathbf{x}\, C^t$ in $\mathsf{Q}^\circ\mathsf{S4.t}$. Translations of the axiom schemes of Definition 4.1 give:

$$(\forall x A \to A(y/x))^t \quad = \Box_F(\Box_F \forall x A^t \to A(y/x)^t)$$

$$(A(y/x) \to \exists x A)^t \quad = \Box_F(A(y/x)^t \to \Diamond_P \exists x A^t)$$

$$(\forall x(A \to B) \to (A \to \forall x B))^t$$

$$= \Box_F(\Box_F \forall x \Box_F(A^t \to B^t) \to \Box_F(A^t \to \Box_F \forall x B^t))$$

$$(\forall x(A \to B) \to (\exists x A \to B))^t$$

$$= \Box_F(\Box_F \forall x \Box_F(A^t \to B^t) \to \Box_F(\Diamond_P \exists x A^t \to B^t))$$

If C is an instance of one of these axiom schemes, then we obtain a proof of $\forall \mathbf{x}\, C^t$ in $\mathsf{Q}^\circ\mathsf{S4.t}$ by Lemma 4.28 and by applying (Gen) to the free variables of C. More precisely, for the first axiom we use (i) of Lemma 4.28 when x is not free in A and (ii) when x is free in A. Similarly, for the second axiom we use (iii) or (iv) of Lemma 4.28. Finally, for the third axiom we use (v) and for the fourth axiom we use (vi) of Lemma 4.28. ☐

Lemma 4.30. *Let A, B be formulas of $\mathcal{L}'$, $\mathbf{x}$ the list of variables free in $A \to B$, $\mathbf{y}$ the list of variables free in A, and $\mathbf{z}$ the list of variables free in B. If* $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall\mathbf{x}(A \to B)^t$ *and* $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall\mathbf{y}A^t$, *then* $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall\mathbf{z}B^t$.

Proof. Let $\mathbf{u}$ be the list of variables free in A but not in B, $\mathbf{v}$ the list of variables free in B but not in A, and $\mathbf{w}$ the list of variables free in both A and B. We then have that $\mathbf{x}$ is the union of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$; $\mathbf{y}$ is the union of $\mathbf{u}$ and $\mathbf{w}$; and $\mathbf{z}$ is the union of $\mathbf{v}$ and $\mathbf{w}$. Thus, we want to show that if $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall\mathbf{u}\,\forall\mathbf{v}\,\forall\mathbf{w}(A \to B)^t$ and $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall\mathbf{u}\,\forall\mathbf{w}A^t$, then $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall\mathbf{v}\,\forall\mathbf{w}B^t$. We have the proof

1. $\forall\mathbf{u}\,\forall\mathbf{v}\,\forall\mathbf{w}\,\Box_F(A^t \to B^t)$
2. $\forall\mathbf{u}\,\forall\mathbf{w}\,\forall\mathbf{v}\,\Box_F(A^t \to B^t)$
3. $\forall\mathbf{u}\,\forall\mathbf{w}\,\forall\mathbf{v}\,(\Box_F A^t \to \Box_F B^t)$
4. $\forall\mathbf{u}\,\forall\mathbf{w}\,(\Box_F A^t \to \forall\mathbf{v}\,\Box_F B^t)$
5. $\forall\mathbf{u}\,\forall\mathbf{w}\,\Box_F A^t \to \forall\mathbf{u}\,\forall\mathbf{w}\,\forall\mathbf{v}\,\Box_F B^t$
6. $\forall\mathbf{u}\,\forall\mathbf{w}A^t$
7. $\forall\mathbf{u}\,\forall\mathbf{w}\,\Box_F A^t$
8. $\forall\mathbf{u}\,\forall\mathbf{w}\,\forall\mathbf{v}\,\Box_F B^t$
9. $\forall\mathbf{u}\,\forall\mathbf{w}\,\forall\mathbf{v}\,B^t$

10 $\forall\mathbf{w}\,\forall\mathbf{v}\,B^t$

11 $\forall\mathbf{v}\,\forall\mathbf{w}\,B^t$

where 1 and 6 are assumptions; 2 and 11 follow from 1 and 10 by switching the order of quantification; 3 is obtained from 2 by distributing $\Box_F$ inside the universal quantifiers; 4 follows from Lemma 4.27(ii) because all the variables in $\mathbf{v}$ are not free in $\Box_F A^t$; 5 is obtained by distributing the universal quantifiers; 7 follows from 6 by introducing $\Box_F$ inside the quantifiers; 8 is obtained by (MP) from 5 and 7; 9 follows from 8 by the T-axiom for $\Box_F$; 10 follows from 9 by NID because no variable in $\mathbf{u}$ is free in B^t. $\square$

Lemma 4.31. *Let A be a formula of $\mathcal{L}'$, x a variable, $\mathbf{y}$ the list of variables free in A, and $\mathbf{z}$ the list of variables free in $\forall x A$. If $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{y} A^t$, then $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}\,(\forall x A)^t$.*

Proof. If x is in $\mathbf{y}$, then without loss of generality we may assume that $\mathbf{y}$ is $\mathbf{z}$ concatenated with x. Therefore, by assumption we have $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}\, \forall x A^t$. If x is not in $\mathbf{y}$, then $\mathbf{y} = \mathbf{z}$. Thus, by (Gen) for x and by switching the order of quantifiers, we again obtain $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}\, \forall x A^t$. We can then introduce $\Box_F$ inside the quantifiers to obtain $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}\, \Box_F \forall x A^t$ which means $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall \mathbf{z}\,(\forall x A)^t$. $\square$

Theorem 4.32. *Let A be a formula of $\mathcal{L}'$ and $x_1, \ldots, x_n$ the free variables of A. If $\mathsf{IQC} \vdash A$, then $\mathsf{Q}^\circ\mathsf{S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$.*

Proof. The proof is by induction on the length of the proof of A in IQC. If A is an instance of an axiom of IQC, then the result follows from Lemma 4.29. Lemma 4.30 takes care of the case in which the last step of the proof of A is an application of (MP). Finally, if the last step of the proof of A is an application of (Gen) to the variable x, use Lemma 4.31. $\square$

Remark 4.33. We are prefixing the translation of A with $\forall x_1 \cdots \forall x_n$ because it is not true in general that $\mathsf{IQC} \vdash A$ implies $\mathsf{Q}^\circ\mathsf{S4.t} \vdash A^t$. For example, if A is an instance of the universal instantiation axiom, which is an axiom of IQC, then A^t is not in general a theorem of $\mathsf{Q}^\circ\mathsf{S4.t}$.

Definition 4.34.

- For an IQC-frame $\mathfrak{F} = (W, R, D)$ let $\overline{\mathfrak{F}} = (W, R, D, U)$ where $U = \bigcup\{D_w \mid w \in W\}$.

- For an IQC-model $\mathfrak{M} = (\mathfrak{F}, I)$ let $\overline{\mathfrak{M}} = (\overline{\mathfrak{F}}, I)$.

Remark 4.35.

- It is obvious that $\overline{\mathfrak{F}}$ is a $\mathsf{Q^\circ S4.t}$-frame.
- If I is an interpretation in $\mathfrak{F}$, then I is also an interpretation in $\overline{\mathfrak{F}}$ because for each n-ary predicate letter P we have $I_w(P) \subseteq D_w^n \subseteq U^n$. Therefore, $\overline{\mathfrak{M}}$ is well defined.
- The w-assignments in $\mathfrak{F}$ are exactly the w-inner assignments in $\overline{\mathfrak{F}}$.

Lemma 4.36. *Let A be a formula of $\mathcal{L}'$, $\mathfrak{M} = (\mathfrak{F}, I)$ a $\mathsf{Q^\circ S4.t}$-model, and σ an assignment in $\mathfrak{F}$. If $v, w \in W$ with vRw, then $\mathfrak{M} \vDash_v^\sigma A^t$ implies $\mathfrak{M} \vDash_w^\sigma A^t$.*

Proof. Suppose vRw and $\mathfrak{M} \vDash_v^\sigma A^t$. By Theorem 4.25 and Lemma 4.26, $\mathfrak{M} \vDash_v^\sigma A^t \to \Box_F A^t$. Therefore, $\mathfrak{M} \vDash_v^\sigma \Box_F A^t$, which yields $\mathfrak{M} \vDash_w^\sigma A^t$ because vRw. ∎

Proposition 4.37. *Let A be a formula of $\mathcal{L}'$, $\mathfrak{M} = (\mathfrak{F}, I)$ an IQC-model based on an IQC-frame $\mathfrak{F} = (W, R, D)$, and $w \in W$.*

1. *For each w-assignment σ, $\mathfrak{M} \vDash_w^\sigma A$ iff $\overline{\mathfrak{M}} \vDash_w^\sigma A^t$.*
2. *If $x_1, \ldots, x_n$ are the free variables of A, then $\mathfrak{M} \vDash_w A$ iff $\overline{\mathfrak{M}} \vDash_w \forall x_1 \cdots \forall x_n A^t$.*

Proof. (i). Induction on the complexity of A. Let A be an atomic formula $P(x_1, \ldots, x_n)$. Since wRv implies $I_w(P) \subseteq I_v(P)$ and R is reflexive, we have

$$\begin{aligned}
\mathfrak{M} \vDash_w^\sigma P(x_1, \ldots, x_n) &\text{ iff } (\sigma(x_1), \ldots, \sigma(x_n)) \in I_w(P) \\
&\text{ iff } (\forall v \in W)(wRv \Rightarrow (\sigma(x_1), \ldots, \sigma(x_n)) \in I_v(P)) \\
&\text{ iff } \overline{\mathfrak{M}} \vDash_w^\sigma \Box_F P(x_1, \ldots, x_n) \\
&\text{ iff } \overline{\mathfrak{M}} \vDash_w^\sigma P(x_1, \ldots, x_n)^t
\end{aligned}$$

The cases where $A = \bot$, $A = B \wedge C$, and $A = B \vee C$ are straightforward.

If $A = B \to C$, then using the inductive hypothesis, we have

$$\begin{aligned}\mathfrak{M} \vDash_w^\sigma B \to C \text{ iff } & (\forall v \in W)(wRv \Rightarrow (\mathfrak{M} \vDash_v^\sigma B \Rightarrow \mathfrak{M} \vDash_v^\sigma C)) \\ \text{iff } & (\forall v \in W)(wRv \Rightarrow (\overline{\mathfrak{M}} \vDash_v^\sigma B^t \Rightarrow \overline{\mathfrak{M}} \vDash_v^\sigma C^t)) \\ \text{iff } & \overline{\mathfrak{M}} \vDash_w^\sigma \Box_F(B^t \to C^t) \\ \text{iff } & \overline{\mathfrak{M}} \vDash_w^\sigma (B \to C)^t.\end{aligned}$$

If $A = \forall x B$, then using the inductive hypothesis, we have

$$\begin{aligned}\mathfrak{M} \vDash_w^\sigma \forall x B \text{ iff } & (\forall v \in W)(wRv \Rightarrow \text{for each } v\text{-assignment } \tau \text{ that is} \\ & \quad \text{an } x\text{-variant of } \sigma \text{ we have } \mathfrak{M} \vDash_v^\tau B) \\ \text{iff } & (\forall v \in W)(wRv \Rightarrow \text{for each assignment } \tau \text{ that is} \\ & \quad \text{an } x\text{-variant of } \sigma \text{ with } \tau(x) \in D_v \text{ we have } \overline{\mathfrak{M}} \vDash_v^\tau B^t) \\ \text{iff } & \overline{\mathfrak{M}} \vDash_w^\sigma \Box_F \forall x B^t \\ \text{iff } & \overline{\mathfrak{M}} \vDash_w^\sigma (\forall x B)^t.\end{aligned}$$

If $A = \exists x B$, then using the inductive hypothesis, reflexivity of R, Lemma 4.36, and the fact that vRw implies $D_v \subseteq D_w$, we have

$$\begin{aligned}\mathfrak{M} \vDash_w^\sigma \exists x B \text{ iff } & \text{there is a } w\text{-assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\ & \quad \text{such that } \mathfrak{M} \vDash_w^\tau B \\ \text{iff } & \text{there is an assignment } \tau \text{ that is an } x\text{-variant of } \sigma \\ & \quad \text{with } \tau(x) \in D_w \text{ such that } \overline{\mathfrak{M}} \vDash_w^\tau B^t\end{aligned}$$

iff there is $v \in W$ such that vRw and an assignment ρ that is

an x-variant of σ with $\rho(x) \in D_v$ such that $\overline{\mathfrak{M}} \vDash_v^\rho B^t$

iff $\overline{\mathfrak{M}} \vDash_w^\sigma \Diamond_P \exists x B^t$

iff $\overline{\mathfrak{M}} \vDash_w^\sigma (\exists x B)^t$.

(ii). By Definition 4.5, $\mathfrak{M} \vDash_w A$ iff $\mathfrak{M} \vDash_w^\sigma A$ for each w-assignment σ. As noted in Remark 4.35, w-assignments in $\mathfrak{F}$ are exactly the w-inner assignments in $\overline{\mathfrak{F}}$. Therefore, by (i), $\mathfrak{M} \vDash_w A$ iff $\overline{\mathfrak{M}} \vDash_w^\sigma A^t$ for each w-inner assignment σ. It follows from the interpretation of the universal quantifier in $\overline{\mathfrak{M}}$ that $\overline{\mathfrak{M}} \vDash_w^\sigma A^t$ for each w-inner assignment σ iff $\overline{\mathfrak{M}} \vDash_w \forall x_1 \cdots \forall x_n A^t$. Thus, $\mathfrak{M} \vDash_w A$ iff $\overline{\mathfrak{M}} \vDash_w \forall x_1 \cdots \forall x_n A^t$. $\square$

Theorem 4.38. *Let A be a formula of $\mathcal{L}'$ and $x_1, \ldots, x_n$ the free variables of A. If* $\mathsf{Q^\circ S4.t} \vdash \forall x_1 \cdots \forall x_n A^t$*, then* $\mathsf{IQC} \vdash A$.

Proof. Suppose $\mathsf{IQC} \nvdash A$. Theorem 4.6 implies that there is an IQC-model $\mathfrak{M}$ such that $\mathfrak{M} \nvDash_w A$ for some world w. By Proposition 4.37(ii), $\overline{\mathfrak{M}} \nvDash_w \forall x_1 \cdots \forall x_n A^t$. Thus, $\mathsf{Q^\circ S4.t} \nvdash \forall x_1 \cdots \forall x_n A^t$ by Theorem 4.25. $\square$

By putting Theorems 4.32 and 4.38 together we arrive at the main result of this section.

Theorem 4.39.

- *For a formula A of $\mathcal{L}'$ and $x_1, \ldots, x_n$ the free variables of A, we have*

$$\mathsf{IQC} \vdash A \text{ iff } \mathsf{Q^\circ S4.t} \vdash \forall x_1 \cdots \forall x_n A^t.$$

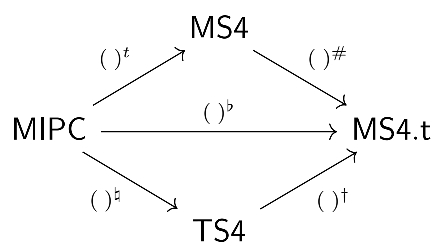
- *If A is a sentence of $\mathcal{L}'$, then*

$$\mathsf{IQC} \vdash A \text{ iff } \mathsf{Q^\circ S4.t} \vdash A^t.$$

Remark 4.40. If we allow constants in $\mathcal{L}'$, Theorem 4.38 is no longer true in its current form. Indeed, constants in IQC and $\mathsf{Q^\circ S4.t}$ behave like free variables and we would have the problem described in Remark 4.33. However, it can be adjusted as follows. Let A be a formula containing free variables $x_1, \dots, x_n$ and constants $c_1, \dots c_m$. If $A(y_1/c_1, \dots, y_m/c_m)$ is the formula obtained by replacing all the constants with fresh variables $y_1, \dots, y_m$, then $\mathsf{IQC} \vdash A$ iff $\mathsf{Q^\circ S4.t} \vdash \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_m A^t(y_1/c_1, \dots, y_m/c_m)$.

4.6 Connections with the monadic case

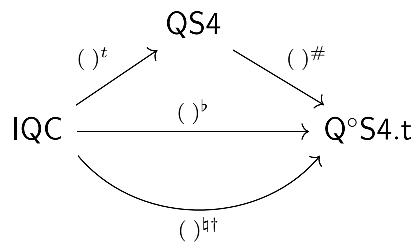
The translation $(-)^\# : \mathsf{MS4} \to \mathsf{MS4.t}$ (see Section 3.4.2) suggests a translation of $\mathsf{QS4}$ into $\mathsf{Q^\circ S4.t}$ which replaces each occurrence of $\Box$ with $\Box_F$. It is easy to see that for sentences this translation is full and faithful. Composing it with the standard Gödel translation of IQC into $\mathsf{QS4}$ yields a translation $\mathsf{IQC} \to \mathsf{Q^\circ S4.t}$ which is different from our temporal translation. This translation restricts to the translation $(-)^{t\#} : \mathsf{MIPC} \to \mathsf{MS4.t}$ for bounded formulas. Thus, the upper part of the following diagram we described at the end of Section 3



extends to the predicate case.

On the other hand, we do not see a natural way to interpret the tense modalities of the logic $\mathsf{TS4}$ defined in Section 3.2 as monadic quantifiers, and hence we cannot think of a natural predicate logic whose monadic fragment is $\mathsf{TS4}$. Thus, the lower part of the diagram does not seem to have a natural extension to the predicate case. Nevertheless, we can consider the predicate analogue of the translation $(-)^{\natural\dagger} : \mathsf{MIPC} \to \mathsf{MS4.t}$. Arguing as in Theorems 3.38 and 3.40 yields a translation of IQC into Q°S4.t that is full and faithful on sentences and coincides, up to logical equivalence in Q°S4.t, with the other two predicate translations we just described.

We thus obtain the following diagram in the predicate setting which is commutative up to logical equivalence in Q°S4.t.



4.7 Open problems and future directions of research

We end Part I by listing several open problems and possible future directions of research.

(1) It is natural to investigate the relationship between the logic MS4.t introduced in Section 3.4.1 and predicate extensions of S4.t. We have that MS4.t is not the monadic fragment of the predicate logic QS4.t. Indeed, as we noted in Section 4.4, the Barcan formula and the converse Barcan formula are both theorems of QS4.t. Thus, the monadic fragment of QS4.t contains both the left commutativity axiom $\Box_F \forall p \to \forall \Box_F p$ and the right commutativity axiom $\forall \Box_F p \to \Box_F \forall p$. On the other hand, it is easy to see (e.g., using the relational semantics for MS4.t defined in Section 3.4.1) that, while MS4.t contains the left commutativity axiom, the right commutativity axiom is not provable in MS4.t. In addition, MS4.t cannot be the monadic fragment of Q°S4.t either since the formula $\forall x A \to A$ is not in general provable in Q°S4.t, whereas $\forall \varphi \to \varphi$ is provable in MS4.t. On the other hand, call a formula φ (in the language of MS4.t) *bounded* if each occurrence of a propositional letter in φ is under the scope of $\forall$. Bounded formulas play the same role as sentences of

$\mathsf{Q^\circ S4.t}$ containing only one fixed variable. It is quite plausible that for a bounded formula φ we have $\mathsf{MS4.t} \vdash \varphi$ iff $\mathsf{Q^\circ S4.t}$ proves the translation of φ where each occurrence of a propositional letter p is replaced with the unary predicate $P(x)$ and $\forall$ is replaced with $\forall x$ (for a similar translation of MIPC and its extensions into IQC and its extensions, see [96]). If true, this would yield that the monadic sentences provable in $\mathsf{Q^\circ S4.t}$ are exactly the bounded formulas φ provable in MS4.t. It would also yield that restricting the temporal translation of IQC into $\mathsf{Q^\circ S4.t}$ to the monadic fragment gives the translation $(-)^\flat : \mathsf{MIPC} \to \mathsf{MS4.t}$ (see Section 3.4.2) for bounded formulas.

(2) It is natural to seek an axiomatization of the full monadic fragment of $\mathsf{Q^\circ S4.t}$. Note that in this fragment $\forall$ does not behave like an S5-modality. For example, $\forall\varphi \to \varphi$ is not in general a theorem of this fragment.

(3) The original proof of McKinsey and Tarski [92, 93] that the Gödel translation of IPC into S4 is full and faithful was algebraic. They proved that the $\Box$-fixpoints of each S4-algebra form a Heyting algebra, and that each Heyting algebra arises this way. In the monadic setting we have that the $\Box$-fixpoints of each MS4-algebra form a monadic Heyting algebra. Fischer-Servi [53] proved that each finite monadic Heyting algebra arises this way. Whether this is true for every monadic Heyting algebra is still an open problem.

(4) The propositional modal logic Grz introduced by Grzegorczyk [70] is obtained by extending the logic S4 with the grz axiom $\Box(\Box(p \to \Box p) \to p) \to p$. Grzegorczyk showed that the Gödel translation is also a full and faithful translation of IPC into Grz. Esakia's theorem [50] states that Grz is the largest propositional modal logic with this property. Moreover, the Blok-Esakia theorem says that the Gödel translation gives rise to a lattice isomorphism between the lattice of propositional intuitionistic logics extending IPC and the

lattice of classical normal modal logics extending Grz (see, e.g, [40, p. 325]). It is natural to ask whether analogous results hold when we replace IPC with MIPC and Grz with the monadic logic MGrz obtained by extending $\mathsf{MS4}$ with the grz axiom. We are currently investigating this direction and the situation is more intricate than in the propositional setting. For instance, Esakia's theorem no longer holds in the monadic setting.

(5) As follows from Theorem 4.25, $\mathsf{Q^\circ S4.t}$ is sound with respect to the class of $\mathsf{Q^\circ S4.t}$-frames. However, its completeness remains an interesting open problem. The standard Henkin construction was modified by Hughes and Cresswell [73] and Corsi [41] to obtain completeness of $\mathsf{Q^\circ K}$. If we adapt their technique to $\mathsf{Q^\circ S4.t}$, we obtain two relations R_F and R_P on the canonical model, one coming from $\Box_F$ and the other from $\Box_P$. There does not seem to be an obvious way to select an appropriate submodel in which the restrictions of these two relations are inverses of each other because the outer domains of accessible worlds are forced to increase by the construction. This problem disappears when constructing the canonical model for $\mathsf{QS4.t}$ because the presence of $\mathsf{BF_F}$ and $\mathsf{CBF_P}$ in each world allows us to select witnesses without expanding the domains of accessible worlds, thus yielding that $\mathsf{QS4.t}$ is sound and complete with respect to the class of $\mathsf{QS4.t}$-frames.

(6) The problem of completeness of $\mathsf{Q^\circ S4.t}$ seems to be closely related to the open problem, stated in [41, p. 1510], of whether $\mathsf{Q^\circ K} + \mathsf{BF}$ is Kripke complete. It appears that answering one of these problems could also provide an answer to the other.

(7) Finally, it is worth investigating translations of intermediate predicate logics into tense predicate logics that are not necessarily extensions of $\mathsf{Q^\circ S4.t}$ (such as the ones considered in [64]). Some such systems admit presheaf semantics which is more general than Kripke semantics.

Part II

Modal operators on rings of continuous functions

5 Modal operators on bounded archimedean ℓ-algebras

In the second part of this thesis we investigate modal operators defined on rings of continuous real-valued functions on compact Hausdorff spaces. The goal of this section is to extend Gelfand duality (also known as Gelfand-Naimark-Stone duality) between uniformly complete bounded archimedean ℓ-algebras and compact Hausdorff spaces investigated in [24] to a duality involving compact Hausdorff spaces endowed with continuous relations. In order to do so, we first observe that a continuous relation on a compact Hausdorff space naturally induces a modal operator on the ring of continuous functions on the space. We then provide an axiomatization of such modal operators and introduce the notion of a modal operator on a bounded archimedean ℓ-algebra A. Conversely, we show that a modal operator on A induces a continuous relation on the dual space of A. This correspondence gives rise to a dual adjunction between the category $\boldsymbol{mba\ell}$ of modal bounded archimedean ℓ-algebras and the category KHF of compact Hausdorff spaces endowed with continuous relations. This dual adjunction restricts to a dual equivalence on the uniformly complete algebras in $\boldsymbol{mba\ell}$. We show that this duality can be thought of as a generalization of the Jónsson-Tarski duality between modal algebras and descriptive frames.

5.1 Gelfand duality

We start by recalling several basic definitions (see [32, Ch. XIII and onwards] or [24]). All rings that we will consider in this thesis are commutative and unital (have multiplicative identity 1).

Definition 5.1. A ring A with a partial order $\leq$ is a *lattice-ordered ring*, or an *ℓ-ring* for short, provided

- $(A, \leq)$ is a lattice;
- $a \leq b$ implies $a + c \leq b + c$ for each c;
- $0 \leq a, b$ implies $0 \leq ab$.

An ℓ-ring A is an *ℓ-algebra* if it is an $\mathbb{R}$-algebra and for each $0 \leq a \in A$ and $0 \leq r \in \mathbb{R}$ we have $0 \leq r \cdot a$.

It is well known and easy to see that the conditions defining ℓ-algebras are equational, and hence ℓ-algebras form a variety. We denote this variety and the corresponding category of ℓ-algebras and unital ℓ-algebra homomorphisms by $\boldsymbol{\ell alg}$.

Definition 5.2. Let A be an ℓ-ring.

- A is *bounded* if for each $a \in A$ there is $n \in \mathbb{N}$ such that $a \leq n \cdot 1$ (that is, 1 is a *strong order unit*).
- A is *archimedean* if for each $a, b \in A$, whenever $n \cdot a \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.

Let $\boldsymbol{ba\ell}$ be the full subcategory of $\boldsymbol{\ell alg}$ consisting of bounded archimedean ℓ-algebras. It is easy to see that $\boldsymbol{ba\ell}$ is not a variety (it is closed under neither products nor homomorphic images).

Let $A \in \boldsymbol{ba\ell}$. For $a \in A$, define the *absolute value* of a by

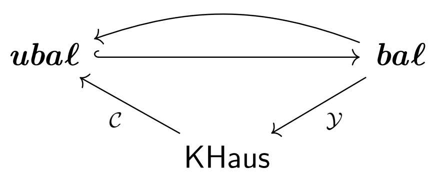
$$|a| = a \vee (-a)$$

and the *norm* of a by

$$\|a\| = \inf\{\lambda \in R \mid |a| \leq \lambda\}.$$ [1]

Then A is *uniformly complete* if the norm is complete. Let $\boldsymbol{ubal}$ be the full subcategory of $\boldsymbol{bal}$ consisting of uniformly complete ℓ-algebras. We also recall from the introduction that KHaus is the category of compact Hausdorff spaces and continuous maps.

Theorem 5.3 (Gelfand duality [62, 105])**.** *There is a dual adjunction between* $\boldsymbol{bal}$ *and* KHaus *which restricts to a dual equivalence between* KHaus *and* $\boldsymbol{ubal}$*.*



The functors $\mathcal{C} : \mathsf{KHaus} \to \boldsymbol{bal}$ and $\mathcal{Y} : \boldsymbol{bal} \to \mathsf{KHaus}$ establishing the dual adjunction are defined as follows. For a compact Hausdorff space X let $\mathcal{C}(X) = C(X)$ be the ring of (necessarily bounded) continuous real-valued functions on X. For a continuous map $\varphi : X \to Y$ let $\mathcal{C}(\varphi) : C(Y) \to C(X)$ be defined by $\mathcal{C}(\varphi)(f) = f \circ \varphi$ for each $f \in C(Y)$. Then $\mathcal{C} : \mathsf{KHaus} \to \boldsymbol{bal}$ is a well-defined contravariant functor.

For $A \in \boldsymbol{bal}$, we recall that an ideal I of A is an *ℓ-ideal* if $|a| \leq |b|$ and $b \in I$ imply $a \in I$, and that ℓ-ideals are exactly the kernels of ℓ-algebra homomorphisms. Let Y_A be the space of maximal ℓ-ideals of A, whose closed sets are exactly sets of the form

$$Z_\ell(I) = \{M \in Y_A \mid I \subseteq M\},$$

where I is an ℓ-ideal of A. The space Y_A is often referred to as the *Yosida space* of A, and it is well known that $Y_A \in \mathsf{KHaus}$ (see [114]). We then set $\mathcal{Y}(A) = Y_A$. For a morphism α

[1]We identify $\lambda \in \mathbb{R}$ with $\lambda \cdot 1 \in A$. If A is nontrivial, we view $\mathbb{R}$ as an ℓ-subalgebra of A.

in $\boldsymbol{ba\ell}$ let $\mathcal{Y}(\alpha) = \alpha^{-1}$. Then $\mathcal{Y} : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ is a well-defined contravariant functor, and the functors $\mathcal{Y}$ and $\mathcal{C}$ yield a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus (see [24, Sec. 3]).

Moreover, for $X \in \mathsf{KHaus}$ we have that $\varepsilon_X : X \to \mathcal{Y}(\mathcal{C}(X))$ is a homeomorphism where

$$\varepsilon_X(x) = \{f \in C(X) \mid f(x) = 0\}.$$

Furthermore, for $A \in \boldsymbol{ba\ell}$ define $\zeta_A : A \to \mathcal{C}(\mathcal{Y}(A))$ by $\zeta_A(a)(M) = \lambda$ where λ is the unique real number satisfying $a + M = \lambda + M$. Then ζ_A is a monomorphism in $\boldsymbol{ba\ell}$ separating points of Y_A. Therefore, by the Stone-Weierstrass theorem, we have:

Proposition 5.4.

1. *The uniform completion of $A \in \boldsymbol{ba\ell}$ is $\zeta_A : A \to C(Y_A)$. Therefore, if A is uniformly complete, then ζ_A is an isomorphism.*

2. *$\boldsymbol{uba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, and the reflector $\zeta : \boldsymbol{ba\ell} \to \boldsymbol{uba\ell}$ assigns to each $A \in \boldsymbol{ba\ell}$ its uniform completion $C(Y_A) \in \boldsymbol{uba\ell}$.*

In the following lemma we collect several facts that will be used subsequently. Its proof can be found in [24, Lem. 2.9].

Lemma 5.5. *Let $\alpha : A \to B$ be a $\boldsymbol{ba\ell}$-morphism.*

1. *$\mathcal{Y}(\alpha)$ is onto iff α is* 1-1 *iff α is a monomorphism.*

2. *$\mathcal{Y}(\alpha)$ is* 1-1 *iff $\alpha[A]$ is uniformly dense in B iff α is an epimorphism.*

3. *$\mathcal{Y}(\alpha)$ is a homeomorphism iff α is a bimorphism.*

5.2 Modal operators on $C(X)$

We now define modal operators on rings of continuous real-valued functions on compact Hausdorff spaces endowed with a continuous relation and study their basic properties. This motivates the definition of a modal operator on $A \in \boldsymbol{ba\ell}$, giving rise to the category $\boldsymbol{mba\ell}$ of modal bounded archimedean ℓ-algebras. We end the section by describing a contravariant functor from KHF to $\boldsymbol{mba\ell}$.

We recall that a *Kripke frame* is a pair $\mathfrak{F} = (X, R)$ where X is a set and R is a binary relation on X (see, e.g, [40, p. 64]). As usual, for $x \in X$ we write

$$R[x] = \{y \in X \mid xRy\} \quad \text{and} \quad R^{-1}[x] = \{y \in X \mid yRx\},$$

and for $U \subseteq X$ we write

$$R[U] = \bigcup\{R[u] \mid u \in U\} \quad \text{and} \quad R^{-1}[U] = \bigcup\{R^{-1}[u] \mid u \in U\}.$$

Definition 5.6. [15] A binary relation R on a compact Hausdorff space X is *continuous* if:

1. $R[x]$ is closed for each $x \in X$.

2. $F \subseteq X$ closed implies $R^{-1}[F]$ is closed.

3. $U \subseteq X$ open implies $R^{-1}[U]$ is open.

If R is a continuous relation on X, we call (X, R) a *compact Hausdorff frame*.

Compact Hausdorff frames are a generalization of both compact Hausdorff spaces and descriptive frames from modal logic (see Definition 5.45). They are related to the Vietoris endofunctor on KHaus.

Definition 5.7. Let $X \in \mathsf{KHaus}$. The Vietoris space $\mathcal{V}(X)$ is the set of closed subsets of X, topologized as follows. If U is an open subset of X, let

$$\Box_U = \{F \in \mathcal{V}(X) \mid F \subseteq U\},$$

$$\Diamond_U = \{F \in \mathcal{V}(X) \mid F \cap U \neq \varnothing\}.$$

The Vietoris topology on $\mathcal{V}(X)$ is the topology with the subbasis

$$\{\Box_U \cap \Diamond_V \mid U, V \text{ open in } X\}.$$

We extend $\mathcal{V}$ to a functor as follows. If $\varphi : X \to Y$ is a continuous function between compact Hausdorff spaces, define $\mathcal{V}(\varphi) : \mathcal{V}(X) \to \mathcal{V}(Y)$ by $\mathcal{V}(\varphi)(F) = \varphi(F)$, the image of F under φ. It is well known that $\mathcal{V}(\varphi)$ is a well-defined continuous map.

It follows from the definition of $\mathcal{V}(X)$ that R is a continuous relation on X iff the corresponding map $\rho_R : X \to \mathcal{V}(X)$ into the Vietoris space of X, given by

$$\rho_R(x) = R[x] = \{y \mid xRy\},$$

is a well-defined continuous map.

Notation 5.8. For a binary relation R on a set X let

$$D = \{x \in X \mid R[x] \neq \varnothing\} = R^{-1}[X],$$

$$E = X \setminus D = \{x \in X \mid R[x] = \varnothing\}.$$

The next lemma is straightforward and we omit the proof.

Lemma 5.9. *If (X, R) is a compact Hausdorff frame, then D and E are both open and closed subsets of X.*

Definition 5.10. For a compact Hausdorff frame (X, R), define $\Box_R$ on $C(X)$ by

$$(\Box_R f)(x) = \begin{cases} \inf fR[x] & \text{if } x \in D \\ 1 & \text{otherwise.} \end{cases}$$

Remark 5.11. We define $\Diamond_R$ by

$$(\Diamond_R f)(x) = \begin{cases} \sup fR[x] & \text{if } x \in D \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\Diamond_R f = 1 - \Box_R(1 - f) \quad \text{and} \quad \Box_R f = 1 - \Diamond_R(1 - f).$$

For, if $x \in D$, then

$$\begin{aligned} 1 - \Box_R(1 - f)(x) &= 1 - \inf\{1 - f(y) \mid xRy\} = 1 - (1 - \sup\{f(y) \mid xRy\}) \\ &= \sup\{f(y) \mid xRy\} = \Diamond_R f(x). \end{aligned}$$

If $x \in E$, then $(1 - \Box_R(1 - f))(x) = 1 - 1 = 0 = (\Diamond_R f)(x)$. Thus, $\Diamond_R f = 1 - \Box_R(1 - f)$, as desired. A similar argument yields $\Box_R f = 1 - \Diamond_R(1 - f)$. Therefore, each of $\Box_R$ and $\Diamond_R$ can be determined from the other.

Remark 5.12. Let (X, R) be a compact Hausdorff frame, $f \in C(X)$, and $x \in X$ with $R[x] \neq \varnothing$. Then $fR[x]$ is a nonempty compact subset of $\mathbb{R}$, and so it has least and greatest elements. Thus, we have

$$(\Box_R f)(x) = \min fR[x] \quad \text{and} \quad (\Diamond_R f)(x) = \max fR[x].$$

Lemma 5.13. *Let (X, R) be a compact Hausdorff frame. If $f \in C(X)$, then $\Box_R f \in C(X)$.*

Proof. To see that $\Box_R f$ is continuous, it is sufficient to show that for each $\lambda \in \mathbb{R}$, both $(\Box_R f)^{-1}(\lambda, \infty)$ and $(\Box_R f)^{-1}(-\infty, \lambda)$ are open in X. We first show that $(\Box_R f)^{-1}(\lambda, \infty)$ is open. Let $x \in X$ and first suppose that $x \in D$. Then $fR[x]$ is a nonempty compact subset of $\mathbb{R}$, so it has a least element. Therefore,

$$\begin{aligned} x \in (\Box_R f)^{-1}(\lambda, \infty) \quad &\text{iff} \quad (\Box_R f)(x) > \lambda \\ &\text{iff} \quad \min(fR[x]) > \lambda \\ &\text{iff} \quad R[x] \subseteq f^{-1}(\lambda, \infty) \\ &\text{iff} \quad x \in X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]. \end{aligned}$$

Next suppose that $x \in E$. Then $(\Box_R f)(x) = 1$. Thus, $E \subseteq (\Box_R f)^{-1}(\lambda, \infty)$ if $\lambda < 1$, and $E \cap (\Box_R f)^{-1}(\lambda, \infty) = \varnothing$ otherwise. Consequently,

$$\begin{aligned} (\Box_R f)^{-1}(\lambda, \infty) &= \big[D \cap (X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)])\big] \cup E \qquad \text{if } \lambda < 1, \text{ and} \\ (\Box_R f)^{-1}(\lambda, \infty) &= D \cap (X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]) \qquad \text{if } 1 \leq \lambda. \end{aligned}$$

Since $f \in C(X)$ and R is continuous, $X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]$ is open. Thus, $(\Box_R f)^{-1}(\lambda, \infty)$ is open by Lemma 5.9.

We next show that $(\Box_R f)^{-1}(-\infty, \lambda)$ is open. If $x \in D$, then

$$\begin{aligned} x \in (\Box_R f)^{-1}(-\infty, \lambda) \quad &\text{iff} \quad (\Box_R f)(x) < \lambda \\ &\text{iff} \quad \min(fR[x]) < \lambda \\ &\text{iff} \quad R[x] \cap f^{-1}(-\infty, \lambda) \neq \varnothing \\ &\text{iff} \quad x \in R^{-1}[f^{-1}(-\infty, \lambda)]. \end{aligned}$$

If $\lambda \leq 1$, then $E \cap (\Box_R f)^{-1}(-\infty, \lambda) = \varnothing$, and if $1 < \lambda$, then $E \subseteq (\Box_R f)^{-1}(-\infty, \lambda)$. Therefore,

$$\begin{aligned} (\Box_R f)^{-1}(-\infty, \lambda) &= D \cap R^{-1}[f^{-1}(-\infty, \lambda)] \qquad \text{if } \lambda \leq 1, \text{ and} \\ (\Box_R f)^{-1}(-\infty, \lambda) &= \big[D \cap (R^{-1}[f^{-1}(-\infty, \lambda)])\big] \cup E \qquad \text{if } \lambda > 1. \end{aligned}$$

Since $f \in C(X)$ and R is continuous, $R^{-1}[f^{-1}(-\infty, \lambda)]$ is open. Consequently, $(\Box_R f)^{-1}(-\infty, \lambda)$ is open by Lemma 5.9. This completes the proof that if $f \in C(X)$, then $\Box_R f \in C(X)$. $\Box$

In the next lemma we describe the properties of $\Box_R$. For this we recall (see, e.g., [24, Rem 2.2]) that if $A \in \boldsymbol{ba\ell}$ and $a \in A$, then the *positive* and *negative* parts of a are defined as

$$a^+ = a \vee 0 \quad \text{and} \quad a^- = -(a \wedge 0) = (-a) \vee 0.$$

Then $a^+, a^- \geq 0$, $a^+ \wedge a^- = 0$, $a = a^+ - a^-$, and $|a| = a^+ + a^-$. This notation is standard (see, e.g., [86, Def. 11.6]).

Lemma 5.14. *Let (X, R) be a compact Hausdorff frame, $f, g \in C(X)$, and $\lambda \in \mathbb{R}$.*

1. $\Box_R(f \wedge g) = \Box_R f \wedge \Box_R g$. *In particular,* $\Box_R$ *is order preserving.*

2. $\Box_R \lambda = \lambda + (1 - \lambda)(\Box_R 0)$. *In particular,* $\Box_R 1 = 1$.

3. $\Box_R(f^+) = (\Box_R f)^+$.

4. $\Box_R(f + \lambda) = \Box_R f + \Box_R \lambda - \Box_R 0$.

5. *If* $0 \leq \lambda$, *then* $\Box_R(\lambda f) = (\Box_R \lambda)(\Box_R f)$.

Proof. (1). For $x \in D$, we have

$$\begin{aligned}
\Box_R(f \wedge g)(x) &= \inf\{(f \wedge g)(y) \mid y \in R[x]\} = \inf\{\min\{f(y), g(y)\} \mid y \in R[x]\} \\
&= \min\{\inf\{f(y) \mid y \in R[x]\}, \inf\{g(y) \mid y \in R[x]\}\} \\
&= \min\{(\Box_R f)(x), (\Box_R g)(x)\} \\
&= (\Box_R f \wedge \Box_R g)(x).
\end{aligned}$$

If $x \in E$, then $\Box_R(f \wedge g)(x) = 1 = (\Box_R f \wedge \Box_R g)(x)$. Thus, $\Box_R(f \wedge g) = \Box_R f \wedge \Box_R g$.

(2). For $x \in D$, if $\mu \in \mathbb{R}$, we have $(\Box_R \mu)(x) = \inf\{\mu \mid y \in R[x]\} = \mu$. From this we see that $(\Box_R \lambda)(x) = \lambda = (\lambda + (1-\lambda)(\Box_R 0))(x)$. If $x \in E$, then $(\Box_R \lambda)(x) = 1 = (\lambda + (1-\lambda)(\Box_R 0))(x)$. Thus, $\Box_R \lambda = \lambda = \lambda + (1-\lambda)(\Box_R 0)$. Setting $\lambda = 1$ yields $\Box_R 1 = 1$.

(3). For $x \in D$, we have

$$(\Box_R(f^+))(x) = \Box_R(f \vee 0)(x) = \inf\{\max\{f(y), 0\} \mid y \in R[x]\}$$

$$= \max\{\inf\{f(y) \mid y \in R[x]\}, 0\} = \max\{\Box_R f(x), 0\}$$

$$= (\Box_R f \vee 0)(x) = (\Box_R f)^+(x).$$

If $x \in E$, then $(\Box_R(f^+))(x) = 1 = (\Box_R f)^+(x)$. Thus, $\Box_R(f^+) = (\Box_R f)^+$.

(4). For $x \in D$, we have

$$\Box_R(f + \lambda)(x) = \inf\{f(y) + \lambda \mid y \in R[x]\}$$

$$= \inf\{f(y) \mid y \in R[x]\} + \lambda$$

$$= \Box_R f(x) + \lambda.$$

On the other hand,

$$(\Box_R f + \Box_R \lambda - \Box_R 0)(x) = (\Box_R f)(x) + (\Box_R \lambda)(x) - (\Box_R 0)(x) = (\Box_R f)(x) + \lambda.$$

Therefore, $\Box_R(f + \lambda)(x) = (\Box_R f + \Box_R \lambda - \Box_R 0)(x)$. If $x \in E$, then $\Box_R(f + \lambda)(x) = 1 = (\Box_R f + \Box_R \lambda - \Box_R 0)(x)$. Thus, $\Box_R(f + \lambda) = \Box_R f + \Box_R \lambda - \Box_R 0$.

(5). Let $0 \leq \lambda$. For $x \in D$, we have

$$(\Box_R \lambda f)(x) = \inf\{\lambda f(y) \mid y \in R[x]\} = \lambda \inf\{f(y) \mid y \in R[x]\}$$

$$= \lambda(\Box_R f)(x) = (\Box_R \lambda)(x)(\Box_R f)(x) = (\Box_R \lambda \Box_R f)(x).$$

If $x \in E$, then $(\Box_R \lambda f)(x) = 1 = (\Box_R \lambda)(\Box_R f)(x)$. Thus, $\Box_R(\lambda f) = (\Box_R \lambda)(\Box_R f)$. $\Box$

Remark 5.15. Lemma 5.14 can be stated dually in terms of $\Diamond_R$ as follows. Let (X, R) be a compact Hausdorff frame, $f, g \in C(X)$, and $\lambda \in \mathbb{R}$.

1. $\Diamond_R(f \vee g) = \Diamond_R f \vee \Diamond_R g$. In particular, $\Diamond_R$ is order preserving.

2. $\Diamond_R \lambda = \lambda(\Diamond_R 1)$. In particular, $\Diamond_R 0 = 0$.

3. $\Diamond_R(f \wedge 1) = (\Diamond_R f) \wedge 1$.

4. $\Diamond_R(f + \lambda) = \Diamond_R f + \Diamond_R \lambda$.

5. If $0 \leq \lambda$, then $\Diamond_R(\lambda f) = \Diamond_R \lambda \Diamond_R f$.

The identities (1), (3), and (5) are direct translations of the corresponding identities for $\Box_R$. However, the identities (2) and (4) are simpler. We next show why $\Diamond_R$ affords such simplifications.

For (2), since $\Diamond_R 1 = 1 - \Box_R 0$, by Lemma 5.14(2),

$$\Diamond_R \lambda = 1 - \Box_R(1 - \lambda) = 1 - (1 - \lambda + \lambda \Box_R 0) = \lambda(1 - \Box_R 0) = \lambda \Diamond_R 1.$$

For (4), by using (4) and (2) of Lemma 5.14, we have

$$\begin{aligned} \Diamond_R(f + \lambda) &= 1 - \Box_R(1 - (f + \lambda)) = 1 - \Box_R((1 - f) - \lambda) \\ &= 1 - (\Box_R(1 - f) + \Box_R(-\lambda) - \Box_R 0) = \Diamond_R f - \Box_R(-\lambda) + \Box_R 0 \\ &= \Diamond_R f - (-\lambda + (1 + \lambda)\Box_R 0) + \Box_R 0 = \Diamond_R f + \lambda(1 - \Box_R 0) \\ &= \Diamond_R f + \lambda \Diamond_R 1 = \Diamond_R f + \Diamond_R \lambda. \end{aligned}$$

In Remark 5.24 we explain why we prefer to work with $\Box_R$.

Lemmas 5.13 and 5.14 motivate the main definition of this section.

Definition 5.16.

1. Let $A \in \boldsymbol{ba\ell}$. We say that a unary function $\Box : A \to A$ is a *modal operator* on A provided $\Box$ satisfies the following axioms for each $a, b \in A$ and $\lambda \in \mathbb{R}$:

   (M1) $\Box(a \wedge b) = \Box a \wedge \Box b$.

   (M2) $\Box \lambda = \lambda + (1 - \lambda)\Box 0$.

   (M3) $\Box(a^+) = (\Box a)^+$.

   (M4) $\Box(a + \lambda) = \Box a + \Box \lambda - \Box 0$.

   (M5) $\Box(\lambda a) = (\Box \lambda)(\Box a)$ provided $\lambda \geq 0$.

2. If $\Box$ is a modal operator on $A \in \boldsymbol{ba\ell}$, then we call the pair $\mathfrak{A} = (A, \Box)$ a *modal bounded archimedean ℓ-algebra*.

3. Let $\boldsymbol{mba\ell}$ be the category of modal bounded archimedean ℓ-algebras and unital ℓ-algebra homomorphisms preserving $\Box$.

Remark 5.17. We can define $\Diamond : A \to A$ dual to $\Box$ by $\Diamond a = 1 - \Box(1 - a)$ for each $a \in A$. Then $(A, \Diamond)$ satisfies the axioms for $\Diamond$ dual to the ones for $\Box$ in Definition 5.16(1) (see Remark 5.15). While algebras in $\boldsymbol{mba\ell}$ can be axiomatized either in the signature of $\Box$ or $\Diamond$, we prefer to work with $\Box$ for the reasons given in Remark 5.24.

Remark 5.18. If $\Box 0 = 0$, then (M2), (M4), and (M5) simplify to the following:

(M2′) $\Box \lambda = \lambda$.

(M4′) $\Box(a + \lambda) = \Box a + \lambda$.

(M5′) $\Box(\lambda a) = \lambda \Box a$ provided $\lambda \geq 0$.

Moreover, (M2′) follows from (M4′) by setting $a = 0$. Furthermore, $\Diamond a = -\Box(-a)$. In Remark 5.44 we will see that $\Box 0 = 0$ holds iff the binary relation $R_\Box$ on the Yosida space of A is serial (see Definition 5.23).

The following technical lemma lists some properties of modal operators on bounded archimedean ℓ-algebras that will be used throughout the section.

Lemma 5.19. *Let* $(A, \Box) \in \boldsymbol{mba\ell}$, $a, b \in A$, *and* $\lambda \in \mathbb{R}$.

1. $a \leq b$ *implies* $\Box a \leq \Box b$.

2. $\Box 1 = 1$.

3. $a \geq 0$ *implies* $\Box a \geq 0$.

4. $(\Box 0)(\Box a) = \Box 0$. *In particular,* $\Box 0$ *is an idempotent.*

5. $\Box(a + \lambda) = \Box a + \lambda(1 - \Box 0)$.

6. $\Diamond a = -\Box(-a)(1 - \Box 0)$.

7. $(\Diamond a)(\Box 0) = 0$.

Proof. (1). If $a \leq b$, then $a \wedge b = a$. Therefore, by (M1), $\Box a = \Box(a \wedge b) = \Box a \wedge \Box b$. Thus, $\Box a \leq \Box b$.

(2). This follows by substituting $\lambda = 1$ in (M2).

(3). From (M3) and $a \geq 0$ we have $\Box a = \Box(a^+) = (\Box a)^+ \geq 0$.

(4). By (M5), $\Box 0 = \Box(0a) = (\Box 0)(\Box a)$. Setting $a = 0$ gives $(\Box 0)^2 = \Box 0$.

(5). By (M4), $\Box(a+\lambda) = \Box a + \Box\lambda - \Box 0$. By (M2), $\Box\lambda = \lambda + (1-\lambda)(\Box 0) = \lambda(1-\Box 0) + \Box 0$. Therefore, $\Box\lambda - \Box 0 = \lambda(1 - \Box 0)$, and so (5) follows.

(6). By (M4), (2), and (4) we have

$$\begin{aligned}\Diamond a &= 1 - \Box(1-a) = 1 - (\Box(-a) + \Box 1 - \Box 0) \\ &= -\Box(-a) + \Box 0 = -\Box(-a) + \Box(-a)\Box 0 \\ &= -\Box(-a)(1 - \Box 0).\end{aligned}$$

(7). Since $\Box 0$ is an idempotent by (4), we have $(1 - \Box 0)\Box 0 = 0$. Multiplying both sides of (6) by $\Box 0$ yields $\Diamond a \Box 0 = 0$. $\Box$

As follows from Lemmas 5.13 and 5.14, if (X, R) is a compact Hausdorff frame, then $(C(X), \Box_R) \in$ ***mbaℓ***. We now extend this correspondence to a contravariant functor. For this we recall the definition of a bounded morphism.

Definition 5.20.

1. A *bounded morphism* (or *p-morphism*) between Kripke frames $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ is a map $f : X \to Y$ satisfying $f(R[x]) = S[f(x)]$ for each $x \in X$ (equivalently, $f^{-1}(S^{-1}[y]) = R^{-1}[f^{-1}(y)]$ for each $y \in Y$).

2. Let KHF be the category of compact Hausdorff frames and continuous bounded morphisms.

Lemma 5.21. *If $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ are compact Hausdorff frames and $\varphi : X \to Y$ is a continuous bounded morphism, then $\mathcal{C}(\varphi)$ is a morphism in* ***mbaℓ****.*

Proof. That $\mathcal{C}(\varphi)$ is a $\boldsymbol{ba\ell}$-morphism follows from Gelfand duality. Therefore, it is sufficient to prove that $\mathcal{C}(\varphi)$ preserves $\Box$; that is, $\mathcal{C}(\varphi)(\Box_S f) = \Box_R \mathcal{C}(\varphi)(f)$ for each $f \in C(Y)$. Since φ is a bounded morphism, $\varphi(R[x]) = S[\varphi(x)]$ for each $x \in X$. Let $x \in X$ and $f \in C(Y)$. If $R[x] \neq \varnothing$, then $S[\varphi(x)] \neq \varnothing$, so

$$\begin{aligned}\mathcal{C}(\varphi)(\Box_S f)(x) &= (\Box_S f \circ \varphi)(x) = (\Box_S f)(\varphi(x)) = \inf(f(S[\varphi(x)])) \\ &= \inf(f(\varphi(R[x]))) = \inf((f \circ \varphi)(R[x])) = \Box_R(f \circ \varphi)(x) \\ &= \Box_R(\mathcal{C}(\varphi)(f))(x).\end{aligned}$$

If $R[x] = \varnothing$, then $S[\varphi(x)] = \varnothing$, so $\mathcal{C}(\varphi)(\Box_S f)(x) = (\Box_S f)(\varphi(x)) = 1 = (\Box_R \mathcal{C}(\varphi)(f))(x)$. Thus, $\mathcal{C}(\varphi)(\Box_S f) = \Box_R \mathcal{C}(\varphi)(f)$. $\Box$

Theorem 5.22. *There is a contravariant functor* $\mathcal{C} : \mathsf{KHF} \to \boldsymbol{mba\ell}$ *which sends* $\mathfrak{F} = (X, R)$ *to* $\mathcal{C}(\mathfrak{F}) = (C(X), \Box_R)$ *and a morphism* φ *in* KF *to* $\mathcal{C}(\varphi)$.

Proof. If $\mathfrak{F} \in \mathsf{KHF}$, then $\mathcal{C}(\mathfrak{F}) \in \boldsymbol{mba\ell}$ by Lemmas 5.13 and 5.14. If φ is a morphism in KHF, then $\mathcal{C}(\varphi)$ is a morphism in $\boldsymbol{mba\ell}$ by Lemma 5.21. It is elementary to see that $\mathcal{C}(\psi \circ \varphi) = \mathcal{C}(\varphi) \circ \mathcal{C}(\psi)$ and that $\mathcal{C}$ preserves identity morphisms. Thus, $\mathcal{C}$ is a contravariant functor. $\Box$

5.3 Continuous relations on the Yosida space

We now define a contravariant functor $\mathcal{Y} : \boldsymbol{mba\ell} \to \mathsf{KHF}$.

Let $A \in \boldsymbol{ba\ell}$. For $S \subseteq A$ let

$$S^+ = \{a \in S \mid a \geq 0\}.$$

We point out that if I is an ℓ-ideal of A, then $I^+ = \{a^+ \mid a \in I\}$.

Definition 5.23. Let $(A, \Box) \in \boldsymbol{mba\ell}$ and let Y_A be the Yosida space of A. Define $R_\Box$ on Y_A by

$$xR_\Box y \quad \text{iff} \quad \Box y^+ \subseteq x \quad \text{iff} \quad y^+ \subseteq \Box^{-1}x.$$

Remark 5.24. We have that $xR_\Box y$ iff $(\forall a \geq 0)(a + y = 0 + y \Rightarrow \Box a + x = 0 + x)$. If we work with $\Diamond$ instead of $\Box$, since $\Diamond a = 1 - \Box(1 - a)$, the definition becomes $xR_\Box y$ iff $(\forall b \leq 1)(b + y = 1 + y \Rightarrow \Diamond b + x = 1 + x)$. Thus, $xR_\Box y$ iff $\{1 - \Diamond b \mid 1 - b \in y, b \leq 1\} \subseteq x$. This more complicated definition is one reason why we prefer to work with $\Box$ rather than $\Diamond$. Another is that, as is standard in working with ordered algebras, using $\Box$ allows us to work with the positive cone rather than the set of elements below 1.

For a topological space X and a continuous real-valued function f on X, we recall (see, e.g., [65, p. 14]) that the *zero set* of f is

$$Z(f) = \{x \in X \mid f(x) = 0\}$$

and the *cozero set* of f is

$$\text{coz}(f) = X \setminus Z(f) = \{x \in X \mid f(x) \neq 0\}.$$

In analogy with the definition above, following [24] we define the *zero set* of an element a of $A \in \boldsymbol{ba\ell}$ as

$$Z_\ell(a) = \{x \in Y_A \mid a \in x\}.$$

If $S \subseteq A$, then we set

$$Z_\ell(S) = \bigcap\{Z_\ell(a) \mid a \in S\} = \{x \in Y_A \mid S \subseteq x\}.$$

It is easy to see that if I is the ℓ-ideal generated by S, then $Z_\ell(S) = Z_\ell(I)$. We define the *cozero set of* S as

$$\mathrm{coz}_\ell(S) = Y_A \setminus Z_\ell(S) = \{x \in Y_A \mid S \nsubseteq x\}.$$

Thus, the family $\{\mathrm{coz}_\ell(a) \mid a \in A\}$ constitutes a basis for the topology on Y_A.

Remark 5.25. Let $A \in \boldsymbol{ba\ell}$, Y_A be the Yosida space of A, $x \in Y_A$, and $a \in A$.

1. x is a prime ideal of A because $A/x \cong \mathbb{R}$. This is a consequence of Hölder's theorem (see, e.g., [72, Cor. 2.7]).

2. Either $a^+ \in x$ or $a^- \in x$. This follows from (1) and $a^+a^- = 0$.

3. $a^+ \in x$ and $a^- \notin x$ iff $a + x < 0 + x$ (see [26, Rem. 2.11]).

4. $a^+ \in x$ iff $a+x \leq 0+x$. For, if $a^+ \in x$, then $a+x = (a^+ - a^-)+x = -a^- + x \leq 0+x$ since $a^- \geq 0$. Conversely, if $a + x \leq 0 + x$, then either $a + x < 0 + x$, in which case $a^+ \in x$ by (3), or $a + x = 0 + x$, in which case $a \in x$, so $a^+ \in x$.

5. $a^- \in x$ and $a^+ \notin x$ iff $a + x > 0 + x$ (see [26, Rem. 2.11]).

6. $a^- \in x$ iff $a+x \geq 0+x$. The proof is similar to that of (4) but uses (5) instead of (3).

Recalling Notation 5.8, if $(Y_A, R_\Box)$ is the dual of $(A, \Box) \in \boldsymbol{mba\ell}$, then we denote $R_\Box^{-1}[Y_A]$ by D_A and $Y_A \setminus D_A$ by E_A.

In the following lemma we list some facts about maximal ℓ-ideals of modal bounded archimedean ℓ-algebras that will be used throughout the section.

Lemma 5.26. *Let $(A, \Box) \in \boldsymbol{mba\ell}$, $a \in A$, $\lambda \in \mathbb{R}$, and $x \in Y_A$.*

1. *If $x \in D_A$, then $\Box 0 \in x$.*

2. *If $\Box 0 \in x$, then $\Box(a+\lambda) + x = (\Box a + \lambda) + x$.*

3. *If $\Box 0 \in x$, then $\Box((a-\lambda)^+) \in x$ iff $(\Box a - \lambda)^+ \in x$.*

4. *If $\Box 0 \in x$, then $\Diamond a + x = -\Box(-a) + x$.*

5. *If $\Box 0 \notin x$, then $1 - \Box a \in x$.*

6. *If $\Diamond a \notin x$, then $\Box 0 \in x$.*

Proof. (1). If $x \in D_A$, then there is y with $xR_\Box y$. Therefore, since $0 \in y^+$, we have $\Box 0 \in x$.

(2). By (M4) and (M2), $\Box(a+\lambda) = \Box a + \lambda - \lambda \Box 0$. Therefore, if $\Box 0 \in x$, then $\Box(a+\lambda)+x = (\Box a + \lambda) + x$.

(3). This follows from (M3), Remark 5.25(4), and (2).

(4). Apply Lemma 5.19(6).

(5). By Lemma 5.19(4), $\Box 0 = (\Box 0)(\Box a)$, so $(\Box 0)(1 - \Box a) = 0 \in x$. Since $\Box 0 \notin x$ and x is a prime ideal, $1 - \Box a \in x$.

(6). By Lemma 5.19(7), $(\Diamond a)(\Box 0) = 0 \in x$. Since x is a prime ideal and $\Diamond a \notin x$, we have $\Box 0 \in x$. □

The main goal of the rest of this section is to show that $R_\Box$ is a continuous relation on Y_A. We first show that the $R_\Box$-image of any point is closed.

Proposition 5.27. *$R_\Box[x]$ is closed for every $x \in Y_A$.*

Proof. We prove that $Y_A \setminus R_\Box[x]$ is open for every $x \in Y_A$. Let $y \notin R_\Box[x]$, so $y^+ \not\subseteq \Box^{-1}x$. Therefore, there is $a \geq 0$ such that $a \in y$ and $\Box a \notin x$. By Lemma 5.19(3), $\Box a \geq 0$, so there is $0 \leq \lambda \in \mathbb{R}$ such that $(\Box a - \lambda) + x > 0 + x$ but $(a - \lambda) + y < 0 + y$. By Remark 5.25(3), $(a-\lambda)^- \notin y$ and $(\Box a - \lambda)^+ \notin x$. Thus, $y \in \text{coz}_\ell((a-\lambda)^-)$, and it remains to show that $\text{coz}_\ell((a-\lambda)^-) \cap R_\Box[x] = \varnothing$. Suppose not. Then there is z such that $xR_\Box z$ and $z \in \text{coz}_\ell((a-\lambda)^-)$. Since z is a prime ideal and $(a-\lambda)^- \notin z$, we have $(a-\lambda)^+ \in z$ (see Remark 5.25(2)). But $xR_\Box z$ means $z^+ \subseteq \Box^{-1}x$, so $\Box 0, \Box((a-\lambda)^+) \in x$. Thus, by (M3) and Lemma 5.26(3), $(\Box a - \lambda)^+ \in x$, hence $(\Box a - \lambda) + x \leq 0 + x$. The obtained contradiction proves that $\text{coz}_\ell((a-\lambda)^-) \cap R_\Box[x] = \varnothing$, completing the proof. □

We now show that the inverse image under $R_\Box$ of a closed subset is closed. We first need some technical lemmas whose proofs are among the most challenging of the thesis.

Lemma 5.28.

1. *Let $X \in$ KHaus and $g, h \in C(X)$. If $Z(g) \subseteq \text{int}\, Z(h)$, then there is $f \in C(X)$ such that $h = gf$.*

2. *Let $A \in$ **bal** and $a, s \in A$. If $Z_\ell(a) \subseteq \text{int}\, Z_\ell(s)$, then there is $f \in C(Y_A)$ such that $\zeta_A(s) = \zeta_A(a)f$ in $C(Y_A)$.*

Proof. (1) This is the first part of [65, Prob. 1D, p. 21].

(2) Observe that for each $t \in A$ we have $Z_\ell(t) = Z(\zeta_A(t))$. Therefore, $Z_\ell(a) \subseteq \text{int}\, Z_\ell(s)$ implies $Z(\zeta_A(a)) \subseteq \text{int}\, Z(\zeta_A(s))$. Now apply (1). □

Lemma 5.29. *Let $(A, \Box) \in$ **mbal**, $x \in Y_A$, $S = (A \setminus \Box^{-1}x)^+$, and $a \in (\Box^{-1}x)^+$.*

1. *$\bigcap\{\text{coz}_\ell(s) \mid s \in S\} = \bigcap\{\overline{\text{coz}_\ell(s)} \mid s \in S\}$ for every $s \in S$.*

2. *$\overline{\text{coz}_\ell(s)} \cap Z_\ell(a) \neq \varnothing$ for every $s \in S$.*

3. The family $\{\overline{\mathrm{coz}_\ell(s)} \cap Z_\ell(a) \mid s \in S\}$ has the finite intersection property.

Proof. (1). The inclusion $\subseteq$ is clear. To prove the reverse inclusion, it is sufficient to prove that for each $s \in S$ there is $t \in S$ such that $\overline{\mathrm{coz}_\ell(t)} \subseteq \mathrm{coz}_\ell(s)$. Since $s \in S$, there is $\varepsilon \in \mathbb{R}$ with $\Box s + x > \varepsilon + x > 0 + x$. Set $t = (s - \varepsilon)^+$. Then $t \geq 0$ and

$$\Box t = \Box(s - \varepsilon)^+ = (\Box(s - \varepsilon))^+$$

by (M3). If $\Box t \in x$, then $\Box(s - \varepsilon) + x \leq 0 + x$, so $\Box s - \varepsilon(1 - \Box 0) + x \leq 0 + x$ by Lemma 5.19(5). We have $\Box 0 \in x$ by Lemma 5.26(5) as $\Box a \in x$, so $\Box s - \varepsilon \leq 0 + x$, and hence $\Box s + x \leq \varepsilon + x$. The obtained contradiction shows $\Box t \notin x$, so $t \in S$. Let $z \in Z_\ell(s)$. Then $z \in \zeta_A(s)^{-1}(-\varepsilon, \varepsilon)$, an open set. But $\zeta_A(s)^{-1}(-\varepsilon, \varepsilon) \subseteq Z_\ell(t)$ by definition of t and Remark 5.25(3), so $Z_\ell(s) \subseteq \mathrm{int}\, Z_\ell(t)$. Thus, $\overline{\mathrm{coz}_\ell(t)} \subseteq \mathrm{coz}_\ell(s)$.

(2). Note that $\overline{\mathrm{coz}_\ell(s)} \cap Z_\ell(a) \neq \varnothing$ means that $Z_\ell(a) \not\subseteq \mathrm{int}\, Z_\ell(s)$. We argue by contradiction. Suppose $Z_\ell(a) \subseteq \mathrm{int}\, Z_\ell(s)$. Then by Lemma 5.28(2), there is $f \in C(Y_A)$ such that $\zeta_A(s) = \zeta_A(a) f$ in $C(Y_A)$. Since $C(Y_A)$ is the uniform completion of A (see Proposition 5.4), there is a sequence $\{b_n\} \subseteq A$ such that $f = \lim \zeta_A(b_n)$. It is well known that multiplication is continuous with respect to the norm, so we have $\lim \zeta_A(ab_n) = \zeta_A(a) f = \zeta_A(s)$. Since $s \in S$, there is $\varepsilon > 0$ such that $\Box s + x > \varepsilon + x$, so $(\Box s - \varepsilon) + x > 0 + x$. There is N such that $||s - ab_N|| < \varepsilon$. Therefore, $s < ab_N + \varepsilon$. Take $0 < \lambda \in \mathbb{R}$ such that $b_N \leq \lambda$. Then $s < \lambda a + \varepsilon$, so by Lemmas 5.19(1), 5.26(2), and axiom (M5),

$$\Box s + x \leq \Box(\lambda a + \varepsilon) + x = (\Box(\lambda a) + \varepsilon) + x = (\Box\lambda\Box a + \varepsilon) + x.$$

But $\Box a \in x$, so $\Box s + x \leq \varepsilon + x$, contradicting $\varepsilon + x < \Box s + x$.

(3). We first show that the intersection of any two members of the family contains another member of the family. Let $s, t \in S$. Then $\Box s, \Box t \notin x$. Since x is a maximal ℓ-ideal, $A/x \cong \mathbb{R}$

is totally ordered, so

$$(\Box s \wedge \Box t) + x = \min\{\Box s + x, \Box t + x\} \neq 0 + x,$$

and hence $\Box s \wedge \Box t \notin x$. By (M1), this shows $\Box(s \wedge t) \notin x$, which gives $s \wedge t \in S$. Since $\mathrm{coz}_\ell(s \wedge t) = \mathrm{coz}_\ell(s) \cap \mathrm{coz}_\ell(t)$, we have:

$$\begin{aligned}(\overline{\mathrm{coz}_\ell(s)} \cap Z_\ell(a)) \cap (\overline{\mathrm{coz}_\ell(t)} \cap Z_\ell(a)) &= \overline{\mathrm{coz}_\ell(s)} \cap \overline{\mathrm{coz}_\ell(t)} \cap Z_\ell(a) \\ &\supseteq \overline{\mathrm{coz}_\ell(s) \cap \mathrm{coz}_\ell(t)} \cap Z_\ell(a) \\ &= \overline{\mathrm{coz}_\ell(s \wedge t)} \cap Z_\ell(a).\end{aligned}$$

Because $s \wedge t \in S$, we have that $\overline{\mathrm{coz}_\ell(s \wedge t)} \cap Z_\ell(a)$ is in the family. An easy induction argument then completes the proof because every element of the family is nonempty by (2). □

Proposition 5.30. *Let* $(A, \Box) \in \boldsymbol{mba\ell}$ *and* $x \in Y_A$. *Then* $(\Box^{-1}x)^+ = \bigcup\{y^+ \mid y \in R_\Box[x]\}$.

Proof. The right-to-left inclusion follows from the definition of $R_\Box$. For the left-to-right inclusion, let $a \in (\Box^{-1}x)^+$. By Lemma 5.29(1),

$$\bigcap\{\mathrm{coz}_\ell(s) \cap Z_\ell(a) \mid s \in S\} = \bigcap\{\overline{\mathrm{coz}_\ell(s)} \cap Z_\ell(a) \mid s \in S\}.$$

By Lemma 5.29(3) and compactness of Y_A, this intersection is nonempty. Therefore, there is $y \in \bigcap\{\mathrm{coz}_\ell(s) \cap Z_\ell(a) \mid s \in S\}$. This means that $a \in y$ and $y \cap S = \varnothing$, so $y^+ \subseteq \Box^{-1}x$. Thus, a is contained in some $y \in R_\Box[x]$, completing the proof. □

Lemma 5.31. *Let* $(A, \Box) \in \boldsymbol{mba\ell}$.

1. $R_\Box^{-1}[Z_\ell(a)] = Z_\ell(\Box a)$ *for every* $0 \leq a \in A$.

2. $D_A = Z_\ell(\Box 0)$.

Proof. (1). Let $x \in R_\Box^{-1}[Z_\ell(a)]$. Then there is $y \in Y_A$ such that $xR_\Box y$ and $a \in y$. Therefore, $a \in y^+ \subseteq \Box^{-1}x$. Thus, $\Box a \in x$, and so $x \in Z_\ell(\Box a)$.

For the other inclusion, let $x \in Z_\ell(\Box a)$. Since $\Box a \in x$ and $\Box a \geq 0$, we have $a \in (\Box^{-1}x)^+$. By Proposition 5.30, there is $y \in Y_A$ such that $xR_\Box y$ and $a \in y$. Thus, $x \in R_\Box^{-1}[Z_\ell(a)]$.

(2). This follows from (1) by setting $a = 0$ and using $Y_A = Z_\ell(0)$. □

We will use Lemma 5.31 to prove that $R_\Box^{-1}[F]$ is closed for each closed subset F of Y_A. For this we require Esakia's lemma, which is an important tool in modal logic (see, e.g., [40, Sec. 10.3]). The original statement is for descriptive frames, but it has a straightforward generalization to the setting of compact Hausdorff frames (see [15, Lem. 2.17]). We call a relation R on a compact Hausdorff space X *point-closed* if $R[x]$ is closed for each $x \in X$.

Lemma 5.32 (Esakia's lemma)**.** *If R is a point-closed relation on a compact Hausdorff space X, then for each nonempty down-directed family $\{F_i \mid i \in I\}$ of closed subsets of X we have*

$$R^{-1}\left[\bigcap\{F_i \mid i \in I\}\right] = \bigcap\{R^{-1}[F_i] \mid i \in I\}.$$

Remark 5.33. Let $(A, \Box) \in \boldsymbol{mba\ell}$ and S be a set of nonnegative elements of A closed under addition. Since $0 \leq a, b \leq a + b$ for each $a, b \in S$, we have $Z_\ell(a + b) \subseteq Z_\ell(a) \cap Z_\ell(b)$. Thus, $\{Z_\ell(a) \mid a \in S\}$ is a down-directed family of closed subsets of Y_A. Then, by Esakia's lemma and Lemma 5.31, we have:

$$\begin{aligned} R_\Box^{-1}[Z_\ell(S)] = R_\Box^{-1}\left[\bigcap\{Z_\ell(a) \mid a \in S\}\right] &= \bigcap\{R_\Box^{-1}[Z_\ell(a)] \mid a \in S\} \\ &= \bigcap\{Z_\ell(\Box a) \mid a \in S\} = Z_\ell(\Box S). \end{aligned}$$

In particular, for an ℓ-ideal I, since $Z_\ell(I) = Z_\ell(I^+)$, we have

$$R_\Box^{-1}Z_\ell(I) = R_\Box^{-1}Z_\ell(I^+) = \bigcap\{Z_\ell(\Box a) \mid a \in I^+\}.$$

Proposition 5.34. *$R_\Box^{-1}[F]$ is closed for every closed subset F of Y_A.*

Proof. Since F is a closed subset of Y_A, there is an ℓ-ideal I such that $F = Z_\ell(I)$. By Remark 5.33,

$$R_\Box^{-1}Z_\ell(I) = \bigcap\{Z_\ell(\Box a) \mid a \in I^+\},$$

which is closed because it is an intersection of closed subsets of Y_A. $\Box$

It now remains to show that the inverse image under $R_\Box$ of an open subset is open. We first need some lemmas.

Lemma 5.35. *If $\Diamond a \in x$ and $xR_\Box y$, then $a^+ \in y$.*

Proof. Suppose that $xR_\Box y$ and $a^+ \notin y$. Then $a + y > 0 + y$, so there is $0 < \lambda \in \mathbb{R}$ such that $a+y = \lambda+y$. Therefore, $\lambda-a \in y$, so $(\lambda-a)^+ \in y$. Since $y^+ \subseteq \Box^{-1}x$, we have $(\Box(\lambda-a))^+ \in x$ by (M3), so $(\lambda + \Box(-a))^+ \in x$ by Lemma 5.26(3). Thus, $(\lambda + \Box(-a)) + x \leq 0 + x$, so $\lambda + x \leq -\Box(-a) + x$, and hence $\lambda + x \leq \Diamond a + x$ by Lemma 5.26(4). Since $\lambda + x > 0 + x$, this shows $\Diamond a \notin x$. $\Box$

Lemma 5.36. *$R_\Box^{-1}[\mathrm{coz}_\ell(a)] = \mathrm{coz}_\ell(\Diamond a)$ for every $0 \leq a \in A$.*

Proof. For the left-to-right inclusion, suppose $x \notin \mathrm{coz}_\ell(\Diamond a)$. Then $\Diamond a \in x$. Consider $y \in R_\Box[x]$. By Lemma 5.35, $a = a^+ \in y$, so $y \notin \mathrm{coz}_\ell(a)$. Therefore, $x \notin R_\Box^{-1}[\mathrm{coz}_\ell(a)]$.

For the right-to-left inclusion, let $x \in \mathrm{coz}_\ell(\Diamond a)$. Then $\Diamond a \notin x$, so $\Box 0 \in x$ by Lemma 5.26(6). Therefore, by Lemma 5.26(4), $0 + x \neq \Diamond a + x = -\Box(-a) + x$, and hence $\Box(-a) \notin x$. Since $-a \leq 0$, we have $\Box(-a) + x \leq \Box 0 + x = 0 + x$. Thus, there is $\lambda \in \mathbb{R}$ with $\lambda < 0$ and $\Box(-a) + x = \lambda + x$, so $\Box(-a) - \lambda \in x$. By Lemma 5.26(3), we have

$$\Box((-a-\lambda)^+) \in x \text{ iff } (\Box(-a) - \lambda)^+ \in x.$$

Consequently, by Proposition 5.30,

$$(-a-\lambda)^+ \in (\Box^{-1}x)^+ = \bigcup\{y^+ \mid y \in R_\Box[x]\}.$$

Hence, there is $y \in R_\Box[x]$ such that $(-a-\lambda)^+ \in y$. This means that $(-a-\lambda)+y \le 0+y$, so $a+y \ge -\lambda+y > 0+y$. Therefore, $a \notin y$, and so $y \in \mathrm{coz}_\ell(a)$. Thus, $x \in R_\Box^{-1}[\mathrm{coz}_\ell(a)]$. $\Box$

Proposition 5.37. *$R_\Box^{-1}[U]$ is open for every open subset U of Y_A.*

Proof. Open subsets of Y_A are of the form $\mathrm{coz}_\ell(I) = \bigcup\{\mathrm{coz}_\ell(a) \mid a \in I\}$ for some ℓ-ideal I. Since $\mathrm{coz}_\ell(I) = \bigcup\{\mathrm{coz}_\ell(a) \mid a \in I,\, a \ge 0\}$ and $R_\Box^{-1}$ commutes with arbitrary unions, by Lemma 5.36, we have

$$\begin{aligned} R_\Box^{-1}\,\mathrm{coz}_\ell(I) &= R_\Box^{-1}\bigcup\{\mathrm{coz}_\ell(a) \mid a \in I,\, a \ge 0\} \\ &= \bigcup\{R_\Box^{-1}\,\mathrm{coz}_\ell(a) \mid a \in I,\, a \ge 0\} \\ &= \bigcup\{\mathrm{coz}_\ell(\Diamond a) \mid a \in I,\, a \ge 0\}, \end{aligned}$$

which is open because it is a union of open subsets of Y_A. $\Box$

Putting Propositions 5.27, 5.34, and 5.37 together yields:

Theorem 5.38. *If $(A, \Box) \in \boldsymbol{mba\ell}$, then $(Y_A, R_\Box) \in \mathsf{KHF}$.*

We finish the section by showing how to extend the object correspondence of Theorem 5.38 to a contravariant functor $\mathcal{Y} : \boldsymbol{mba\ell} \to \mathsf{KHF}$.

Lemma 5.39. *Let $(A, \Box), (B, \Box) \in \boldsymbol{mba\ell}$ and $\alpha : A \to B$ be a morphism in $\boldsymbol{mba\ell}$. Then $\mathcal{Y}(\alpha) : Y_B \to Y_A$ is a bounded morphism.*

Proof. For each $y \in Y_A$, we have that y^+ and $\alpha(y^+)$ are sets of nonnegative elements closed under addition, so Remark 5.33 applies. Therefore, since $Z(y^+) = \{y\}$,

$$(\mathcal{Y}(\alpha))^{-1}(R_\Box^{-1}[y]) = (\mathcal{Y}(\alpha))^{-1}(R_\Box^{-1}[Z_\ell(y^+)]) = (\mathcal{Y}(\alpha))^{-1}(Z_\ell(\Box y^+))$$

and

$$Z_\ell(\Box\alpha(y^+)) = R_\Box^{-1}[Z_\ell(\alpha(y^+))].$$

The definition of $\mathcal{Y}(\alpha)$ shows that $(\mathcal{Y}(\alpha))^{-1}(Z_\ell(\Box y^+)) = Z_\ell(\alpha(\Box y^+))$ and $(\mathcal{Y}(\alpha))^{-1}(Z_\ell(y^+)) = Z_\ell(\alpha(y^+))$. This yields

$$(\mathcal{Y}(\alpha))^{-1}(R_\Box^{-1}[y]) = (\mathcal{Y}(\alpha))^{-1}(Z_\ell(\Box y^+)) = Z_\ell(\alpha(\Box y^+))$$

and

$$R_\Box^{-1}[(\mathcal{Y}(\alpha))^{-1}(y)] = R_\Box^{-1}[(\mathcal{Y}(\alpha))^{-1}(Z_\ell(y^+))] = R_\Box^{-1}[Z_\ell(\alpha(y^+))] = Z_\ell(\Box\alpha(y^+)).$$

Consequently, since α commutes with $\Box$, we have $(\mathcal{Y}(\alpha))^{-1}(R_\Box^{-1}[y]) = R_\Box^{-1}[(\mathcal{Y}(\alpha))^{-1}(y)]$, which proves that $\mathcal{Y}(\alpha)$ is a bounded morphism. $\Box$

Putting Theorem 5.38 and Lemma 5.39 together and remembering that $\mathcal{Y} : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ is a contravariant functor yields:

Theorem 5.40. $\mathcal{Y} : \boldsymbol{mba\ell} \to \mathsf{KHF}$ *is a contravariant functor.*

5.4 Duality

We are now ready to prove our main results. We show that $\mathcal{Y}$ and $\mathcal{C}$ yield a dual adjunction between $\boldsymbol{mba\ell}$ and KHF which restricts to a dual equivalence between the category of uniformly complete members of $\boldsymbol{mba\ell}$ and KHF.

Definition 5.41. Let $\boldsymbol{muba\ell}$ be the full subcategory of $\boldsymbol{mba\ell}$ consisting of uniformly complete objects of $\boldsymbol{mba\ell}$.

Proposition 5.42. $\boldsymbol{muba\ell}$ *is a reflective subcategory of* $\boldsymbol{mba\ell}$.

Proof. By Proposition 5.4(2), $\boldsymbol{ubal}$ is a reflective subcategory of $\boldsymbol{bal}$, where $\zeta : \boldsymbol{bal} \to \boldsymbol{ubal}$ is the reflector. We first show that ζ_A is an $\boldsymbol{mbal}$-morphism for each $(A, \Box) \in \boldsymbol{mbal}$. Let $x \in Y_A$. Recall that

$$(\Box_{R_\Box} \zeta_A(a))(x) = \begin{cases} \inf\{\zeta_A(a)(y) \mid xR_\Box y\} & \text{if } x \in D_A \\ 1 & \text{otherwise.} \end{cases}$$

If $x \in E_A$, then $\Box 0 \notin x$ by Lemma 5.31(2). Therefore, $\Box a - 1 \in x$ by Lemma 5.26(5), and hence $\zeta_A(\Box a)(x) = 1 = (\Box_{R_\Box} \zeta_A(a))(x)$. Now let $x \in D_A$. Then $(\Box_{R_\Box} \zeta_A(a))(x) = \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$. We first show that $\zeta_A(\Box a)(x) \le \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$. Suppose that $xR_\Box y$, so $y^+ \subseteq \Box^{-1}x$. Let $\lambda = \zeta_A(a)(y)$. Then $a - \lambda \in y$, so $(a - \lambda)^+ \in y^+ \subseteq \Box^{-1}x$, and hence $(\Box a - \lambda)^+ \in x$ iff $\Box((a - \lambda)^+) \in x$ by Lemma 5.26(3). Therefore,

$$0 = \zeta_A((\Box a - \lambda)^+)(x) = \max\{\zeta_A(\Box a)(x) - \lambda, 0\},$$

so $\zeta_A(\Box a)(x) - \lambda \le 0$, and hence $\zeta_A(\Box a)(x) \le \lambda = \zeta_A(a)(y)$. Thus,

$$\zeta_A(\Box a)(x) \le \inf\{\zeta_A(a)(y) \mid xR_\Box y\}.$$

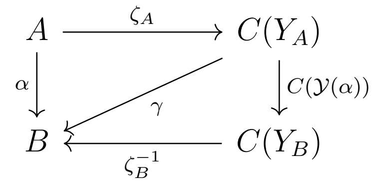
We next show that $\zeta_A(\Box a)(x) \ge \inf\{\zeta_A(a)(y) \mid xR_\Box y\}$. Let $\mu = \zeta_A(\Box a)(x)$. We have $\Box((a - \mu)^+) \in x$ iff $(\Box a - \mu)^+ \in x$. Therefore, by Proposition 5.30,

$$(a - \mu)^+ \in (\Box^{-1}x)^+ = \bigcup\{y^+ \mid xR_\Box y\}.$$

So there is $y \in R_\Box[x]$ such that $(a - \mu)^+ \in y$. Thus, $\max\{\zeta_A(a)(y) - \mu, 0\} = 0$. This yields $\zeta_A(a)(y) - \mu \le 0$, and so $\zeta_A(a)(y) \le \mu = \zeta_A(\Box a)(x)$. Consequently,

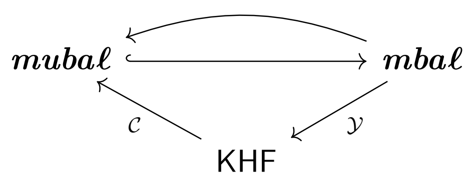
$$\inf\{\zeta_A(a)(y) \mid y \in R_\Box[x]\} \le \zeta_A(\Box a)(x).$$

Next, let $\alpha : A \to B$ be an $\boldsymbol{mba\ell}$-morphism with $B \in \boldsymbol{muba\ell}$. Since α is a $\boldsymbol{ba\ell}$-morphism, there is a unique $\boldsymbol{ba\ell}$-morphism $\gamma : C(Y_A) \to B$, given by $\gamma = \zeta_B^{-1} \circ C(\mathcal{Y}(\alpha))$, such that $\gamma \circ \zeta_A = \alpha$.



As we saw in the paragraph above, ζ_B is an $\boldsymbol{mba\ell}$-morphism. Also, $C(\mathcal{Y}(\alpha)) : C(Y_A) \to C(Y_B)$ is an $\boldsymbol{mba\ell}$-morphism by Lemmas 5.39 and 5.21. Therefore, γ is an $\boldsymbol{mba\ell}$-morphism, concluding the proof. □

Theorem 5.43. *The functors $\mathcal{Y} : \boldsymbol{mba\ell} \to \mathsf{KHF}$ and $\mathcal{C} : \mathsf{KHF} \to \boldsymbol{mba\ell}$ yield a dual adjunction of the categories, which restricts to a dual equivalence between $\boldsymbol{muba\ell}$ and KHF.*



Proof. By Gelfand duality, the functors $\mathcal{Y} : \boldsymbol{ba\ell} \to \mathsf{KHaus}$ and $\mathcal{C} : \mathsf{KHaus} \to \boldsymbol{ba\ell}$ yield a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus that restricts to a dual equivalence between $\boldsymbol{uba\ell}$ and KHaus. The natural transformations are given by $\zeta : 1_{\boldsymbol{ba\ell}} \to \mathcal{C} \circ \mathcal{Y}$ and $\varepsilon : 1_{\mathsf{KHaus}} \to \mathcal{Y} \circ \mathcal{C}$ where we recall from Section 5.1 that $\varepsilon_X : X \to X_{C(X)}$ is defined by

$$\varepsilon_X(x) = M_x = \{f \in C(X) \mid f(x) = 0\}.$$

Therefore, it is sufficient to show that ζ_A is a morphism in $\boldsymbol{mba\ell}$ for each $(A, \Box) \in \boldsymbol{mba\ell}$ and that ε_X is a bounded morphism for each $(X, R) \in \mathsf{KHF}$. We showed in the proof of Proposition 5.42 that $\zeta_A(\Box a) = \Box_{R_\Box} \zeta_A(a)$ for each $(A, \Box) \in \boldsymbol{mba\ell}$ and $a \in A$. Thus, ζ_A is a morphism in $\boldsymbol{mba\ell}$, and hence it remains to show that xRy iff $\varepsilon_X(x) R_{\Box_R} \varepsilon_X(y)$ for each $(X, R) \in \mathsf{KHF}$.

To see this recall that $\varepsilon_X(x) R_{\Box_R} \varepsilon_X(y)$ means that $M_y^+ \subseteq \Box_R^{-1} M_x$. First suppose that xRy and $f \in M_y^+$. Then $f(y) = 0$ and $f \geq 0$. We have $(\Box_R f)(x) = \inf\{f(z) \mid xRz\} = 0$. Therefore, $\Box_R f \in M_x$, and so $f \in \Box_R^{-1} M_x$. This gives $M_y^+ \subseteq \Box_R^{-1} M_x$. Next suppose that $x \not R y$, so $y \notin R[x]$. If $R[x] = \varnothing$, then $(\Box_R 0)(x) = 1$, so $0 \in M_y^+$ but $\Box_R 0 \notin M_x$, yielding $M_y^+ \not\subseteq \Box_R^{-1} M_x$. On the other hand, if $R[x] \neq \varnothing$, since $R[x]$ is closed, by Urysohn's lemma there is $f \geq 0$ such that $f(y) = 0$ and $f(R[x]) = \{1\}$. Thus, $f \in M_y^+$ and $\Box_R f \notin M_x$. Consequently, $M_y^+ \not\subseteq \Box_R^{-1} M_x$. $\Box$

Remark 5.44. In [20, Sec. 5.2] we develop the first steps towards the correspondence theory for $\boldsymbol{mba\ell}$. Namely, we characterize algebraically what it takes for the relation $R_\Box$ on Y_A to satisfy additional first-order properties. We have the following results:

1. $R_\Box$ is serial (i.e. $R_\Box[x] \neq \varnothing$ for each $x \in Y_A$) iff $\Box 0 = 0$ in A.

2. $R_\Box$ is reflexive iff $\Box a \leq a$ for each $a \in A$.

3. $R_\Box$ is transitive iff $\Box a \leq \Box(\Box a(1 - \Box 0) + a \Box 0)$ for each $a \in A$.

4. $R_\Box$ is symmetric iff $\Diamond \Box a(1 - \Box 0) \leq a(1 - \Box 0)$ for each $a \in A$.

In the serial setting, the axioms corresponding to transitivity and symmetry simplify to $\Box a \leq \Box\Box a$ and $\Diamond \Box a \leq a$, which are standard transitivity and symmetry axioms in modal logic. It would be natural to develop the correspondence theory for $\boldsymbol{mba\ell}$ by generalizing these results, with the final goal towards a Sahlqvist type correspondence.

5.5 Connections with modal algebras and descriptive frames

Theorem 5.43 generalizes Gelfand duality. We now show that it also generalizes Jónsson-Tarski duality between modal algebra and descriptive frames. We first recall some definitions.

Definition 5.45.

1. A *modal algebra* is a pair $\mathfrak{A} = (A, \Box)$ where A is a boolean algebra and $\Box$ is a unary function on A preserving finite meets (including 1). The category of modal algebras and modal homomorphisms (boolean homomorphisms preserving $\Box$) is denoted by MA.

2. A compact Hausdorff space is called a *Stone space* if its clopen subsets (i.e. the subsets that are open and closed at the same time) form a basis.

3. A *descriptive frame* is a pair $\mathfrak{F} = (X, R)$ where X is a Stone space and R is a continuous relation on X. The category DF is the full subcategory of KHF whose objects are the descriptive frames.

As we already pointed out, Stone duality generalizes to the following duality:

Theorem 5.46 (Jónsson-Tarski duality [48, 68])**.** MA *is dually equivalent to* DF.

The contravariant functors $(-)^* : \mathsf{DF} \to \mathsf{MA}$ and $(-)_* : \mathsf{MA} \to \mathsf{DF}$ establishing this dual equivalence are defined as follows. For a descriptive Kripke frame $\mathfrak{F} = (X, R)$ let $\mathfrak{F}^* = (\mathsf{Clop}(X), \Box_R)$ where $\mathsf{Clop}(X)$ is the boolean algebra of clopen subsets of X and $\Box_R U = X \setminus R^{-1}[X \setminus U]$ (alternatively, $\Diamond_R U = R^{-1}[U]$). For a bounded morphism f let $f^* = f^{-1}$. Then $(-)^* : \mathsf{DF} \to \mathsf{MA}$ is a well-defined contravariant functor.

For a modal algebra $\mathfrak{A} = (A, \Box)$ let $\mathfrak{A}_* = (X_A, R_\Box)$ where X_A is the set of ultrafilters of A and

$$xR_\Box y \quad \text{iff} \quad (\forall a \in A)(\Box a \in x \Rightarrow a \in y) \quad \text{iff} \quad \Box^{-1}x \subseteq y$$

(alternatively, $xR_\Box y$ iff $(\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$ iff $y \subseteq \Diamond^{-1}x$). For a modal algebra homomorphism h let $h_* = h^{-1}$. Then $(-)_* : \mathsf{MA} \to \mathsf{DF}$ is a well-defined contravariant functor, and the functors $(-)_*$ and $(-)^*$ yield Jónsson-Tarski duality between MA and DF.

To define a functor from $\boldsymbol{mba\ell}$ to MA we recall that for each commutative ring A with 1, the idempotents of A form a boolean algebra $\mathrm{Id}(A)$, where the boolean operations on $\mathrm{Id}(A)$ are defined as follows:

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad \neg e = 1 - e.$$

We point out that if $A \in \boldsymbol{ba\ell}$, then the lattice operations on A restrict to those on $\mathrm{Id}(A)$.

Remark 5.47. We will use the following two identities of f-rings (see [32, Sec. XIII.3] and [32, Cor. XVII.5.1]):

$$(a \wedge b) + c = (a + c) \wedge (b + c) \quad \text{and} \quad (a \wedge b)d = (ad) \wedge (bd) \text{ for } d \geq 0.$$

Lemma 5.48. *If $(A, \Box) \in \boldsymbol{mba\ell}$, then $\Box$ sends idempotents to idempotents.*

Proof. First observe that $e \in A$ is an idempotent iff $1 \wedge 2e = e$. To see this, if e is an idempotent, by Remark 5.47,

$$(1 \wedge 2e) - e = (1 - e) \wedge e = \neg e \wedge e = 0.$$

Therefore, $1 \wedge 2e = e$. Conversely, suppose that $1 \wedge 2e = e$. Then $(1 - e) \wedge e = 0$ by the same calculation. Since each $A \in \boldsymbol{ba\ell}$ is an f-ring (see, e.g., [32, Lem. XVII.5.2]), from

$(1-e)\wedge e = 0$ it follows that $(1-e)e = 0$ (see, e.g., [32, Lem. XVII.5.1]). Thus, $e^2 = e$, and hence e is an idempotent.

For each $a \in A$, by (M5), (M2), and Lemma 5.19(4) we have

$$\Box(2a) = \Box 2 \Box a = (2 - \Box 0)\Box a = (2 - 2\Box 0 + \Box 0)\Box a = 2\Box a(1 - \Box 0) + \Box 0.$$

By Lemma 5.19(3), $\Box 0 \geq 0$, so Lemma 5.19(4) and Remark 5.47 imply

$$(1 \wedge 2\Box a)\Box 0 = \Box 0 \wedge 2\Box a \Box 0 = \Box 0 \wedge 2\Box 0 = \Box 0.$$

Now suppose e is an idempotent, so $e = 1 \wedge 2e$. Since $\Box 0 \leq \Box 1 = 1$, we have $1 - \Box 0 \geq 0$. Thus, by Remark 5.47 and the two identities just proved,

$$\begin{aligned}
\Box e &= \Box(1 \wedge 2e) = 1 \wedge \Box(2e) \\
&= ((1 - \Box 0) + \Box 0) \wedge \Box(2e) \\
&= ((1 - \Box 0) + \Box 0) \wedge (2\Box e(1 - \Box 0) + \Box 0) \\
&= ((1 - \Box 0) \wedge 2\Box e(1 - \Box 0)) + \Box 0 \\
&= (1 \wedge 2\Box e)(1 - \Box 0) + \Box 0 \\
&= (1 \wedge 2\Box e)(1 - \Box 0) + (1 \wedge 2\Box e)\Box 0 \\
&= 1 \wedge 2\Box e.
\end{aligned}$$

Therefore, $\Box e$ is idempotent. □

Lemma 5.49. *If* $(A, \Box) \in \boldsymbol{mba\ell}$*, then* $(\mathrm{Id}(A), \Box) \in \mathsf{MA}$.

Proof. Since $A \in \boldsymbol{ba\ell}$, we have that $\mathrm{Id}(A)$ is a boolean algebra. By Lemma 5.48, $\Box$ is well defined on $\mathrm{Id}(A)$. That $\Box$ preserves finite meets in $\mathrm{Id}(A)$ follows from (M1) and Lemma 5.19(2). Thus, $(\mathrm{Id}(A), \Box) \in \mathsf{MA}$. □

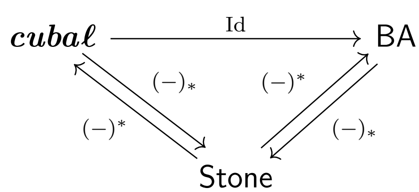
Define $\mathrm{Id} : \boldsymbol{mba\ell} \to \mathsf{MA}$ by sending $(A, \Box) \in \boldsymbol{mba\ell}$ to $(\mathrm{Id}(A), \Box) \in \mathsf{MA}$ and a morphism $A \to B$ in $\boldsymbol{mba\ell}$ to its restriction $\mathrm{Id}(A) \to \mathrm{Id}(B)$. The next lemma is an easy consequence of Lemma 5.49.

Lemma 5.50. $\mathrm{Id} : \boldsymbol{mba\ell} \to \mathsf{MA}$ *is a well-defined covariant functor.*

We recall (see [90] and the references therein) that a commutative ring A is *clean* if each element is the sum of an idempotent and a unit.

Definition 5.51. Let $\boldsymbol{cuba\ell}$ be the full subcategory of $\boldsymbol{uba\ell}$ consisting of those $A \in \boldsymbol{uba\ell}$ where A is clean.

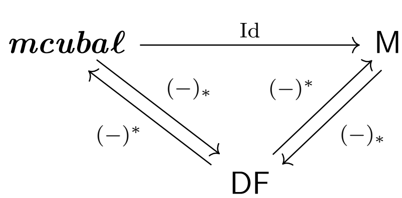
Remark 5.52. By Stone duality for boolean algebras and [24, Prop. 5.20], the following diagram commutes (up to natural isomorphism), and the functor Id yields an equivalence of $\boldsymbol{cuba\ell}$ and BA.



Definition 5.53. Let $\boldsymbol{mcuba\ell}$ be the full subcategory of $\boldsymbol{muba\ell}$ consisting of those $(A, \Box) \in \boldsymbol{muba\ell}$ where A is clean.

As a corollary of Theorems 5.43, 5.46 and Remark 5.52, we obtain:

Theorem 5.54. *The diagram below commutes* (*up to natural isomorphism*) *and the functor* Id *yields an equivalence of* $\boldsymbol{mcuba\ell}$ *and* MA.



This shows that the dual equivalence between $\boldsymbol{mcuba\ell}$ and DF obtained by restricting the duality stated in Theorem 5.43 is the ring-theoretic analogue of Jónsson-Tarski duality. Therefore, we can think of the dual equivalence of Theorem 5.43 as an extension of Jónsson-Tarski duality.

6 The Vietoris functor and modal operators on rings of continuous functions

In this last section of the thesis we provide an alternate, more categorical treatment of the results of the previous section. The Vietoris endofunctor $\mathcal{V} : \mathsf{KHaus} \to \mathsf{KHaus}$ restricts to an endofunctor $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$ on the category of Stone spaces. It is well known that the category DF of descriptive frames (see Definition 5.45) is isomorphic to the category $\mathsf{Coalg}(\mathcal{V})$ of coalgebras for the Vietoris endofunctor $\mathcal{V}$ on Stone (for the definitions of algebra and coalgebra for an endofunctor see Definitions 6.20 and 6.32). Abramsky [1] and Kupke, Kurz, and Venema [84] defined the dual endofunctor $\mathcal{H}$ on the category BA of boolean algebras and showed that the category $\mathsf{Alg}(\mathcal{H})$ of algebras for $\mathcal{H}$ is isomorphic to MA. They obtained as a consequence that the Stone duality between BA and Stone lifts to a dual equivalence between $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$. This yields an elegant new proof of Jónsson-Tarski duality. The isomorphism between DF and the category of coalgebras for $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$ extends to an isomorphism between KHF and the category of coalgebras for $\mathcal{V} : \mathsf{KHaus} \to \mathsf{KHaus}$. We introduce an endofunctor $\mathcal{H}$ on the category $\boldsymbol{ba\ell}$ of bounded archimedean ℓ-algebras and show that there is a dual adjunction between the category $\mathsf{Alg}(\mathcal{H})$ of algebras for $\mathcal{H}$ and the category $\mathsf{Coalg}(\mathcal{V})$ of coalgebras for the Vietoris endofunctor $\mathcal{V}$ on the category of compact Hausdorff spaces. In order to define $\mathcal{H}$ we need to investigate the existence of free objects in $\boldsymbol{ba\ell}$. We also show that Gelfand duality lifts to a dual equivalence between $\mathsf{Coalg}(\mathcal{V})$ and a full reflective subcategory $\mathsf{Alg}^u(\mathcal{H})$ of $\mathsf{Alg}(\mathcal{H})$. Then the dual adjunction between KHF and $\boldsymbol{mba\ell}$ and the dual equivalence between KHF and $\boldsymbol{muba\ell}$ obtained in the previous section follow from the fact that $\mathsf{Coalg}(\mathcal{V})$ and $\mathsf{Alg}(\mathcal{H})$ are isomorphic to KHF and

$\boldsymbol{mba\ell}$, respectively. We show that also $\mathsf{Alg}^u(\mathcal{H})$ can be thought of as a category of algebras by introducing the endofunctor $\mathcal{H}^u$ on $\boldsymbol{uba\ell}$ and showing that $\mathsf{Alg}(\mathcal{H}^u)$ is isomorphic to $\mathsf{Alg}^u(\mathcal{H})$. We conclude the section by showing how our results connect with those from [84] for the category of coalgebras of the Vietoris endofunctor on the category of Stone spaces. We end by listing some possible future research topics and open problems related to the topics covered in the second part of the thesis.

6.1 Free objects in $\boldsymbol{ba\ell}$

Our aim is to generalize the endofunctor $\mathcal{H} : \mathsf{BA} \to \mathsf{BA}$ that is the algebraic counterpart of $\mathcal{V} : \mathsf{Stone} \to \mathsf{Stone}$ to an endofunctor $\mathcal{H} : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$ so that it is the algebraic counterpart of $\mathcal{V} : \mathsf{KHaus} \to \mathsf{KHaus}$. The construction of $\mathcal{H} : \mathsf{BA} \to \mathsf{BA}$ utilizes the existence of free boolean algebras. Thus, if we want to replicate such a construction for $\boldsymbol{ba\ell}$, we need to investigate the existence of free objects in $\boldsymbol{ba\ell}$.

As we pointed out in Section 5.1, $\boldsymbol{\ell alg}$ is a variety, hence has free algebras by Birkhoff's theorem (see, e.g., [38, Thm. 10.12]). Since $\boldsymbol{ba\ell}$ is not a subvariety of $\boldsymbol{\ell alg}$, it does not follow immediately that $\boldsymbol{ba\ell}$ has free algebras. In fact, we show that free algebras on sets do not exist in $\boldsymbol{ba\ell}$. In other words, we show that the forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Sets}$ does not have a left adjoint.

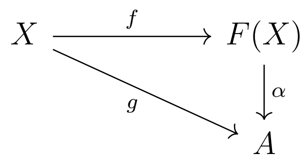
Lemma 6.1. *Let $A, B \in \boldsymbol{ba\ell}$ and $\alpha : A \to B$ be a $\boldsymbol{ba\ell}$-morphism. Then for each $a \in A$ we have $\alpha(|a|) = |\alpha(a)|$ and $\|\alpha(a)\| \leq \|a\|$.*

Proof. Let $a \in A$. Then $\alpha(|a|) = \alpha(a \vee -a) = \alpha(a) \vee -\alpha(a) = |\alpha(a)|$. Since $|a| \leq \|a\|$ and $\alpha(r) = r$ for each $r \in \mathbb{R}$, we have $\alpha(|a|) \leq \alpha(\|a\|) = \|a\|$. Therefore, $|\alpha(a)| = \alpha(|a|) \leq \|a\|$

and hence $\|\alpha(a)\| \leq \|a\|$. $\square$

Theorem 6.2. *The forgetful functor* $U : \boldsymbol{ba\ell} \to \mathsf{Sets}$ *does not have a left adjoint.*

Proof. If U has a left adjoint, then for each $X \in \mathsf{Sets}$, there is $F(X) \in \boldsymbol{ba\ell}$ and a function $f : X \to F(X)$ such that for each $A \in \boldsymbol{ba\ell}$ and each function $g : X \to A$ there is a unique $\boldsymbol{ba\ell}$-morphism $\alpha : F(X) \to A$ satisfying $\alpha \circ f = g$.



Let X be a nonempty set. Pick $x \in X$, choose $r \in \mathbb{R}$ with $r > \|f(x)\|$, and define $g : X \to \mathbb{R}$ by setting $g(y) = r$ for each $y \in X$. There is a (unique) $\boldsymbol{ba\ell}$-morphism $\alpha : F(X) \to \mathbb{R}$ with $\alpha \circ f = g$, so $\alpha(f(x)) = r$. But if $a \in F(X)$, then $\|\alpha(a)\| \leq \|a\|$ by Lemma 6.1. Therefore,

$$r = \|\alpha(f(x))\| \leq \|f(x)\| < r.$$

The obtained contradiction proves that $F(X)$ does not exist. Thus, U does not have a left adjoint. $\square$

The key reason for nonexistence of a left adjoint to the forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Sets}$ can be explained as follows. The norm on A provides a weight function on the set A, and each $\boldsymbol{ba\ell}$-morphism α respects this weight function due to the inequality $\|\alpha(a)\| \leq \|a\|$. The forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{Sets}$ forgets this, which is the obstruction to the existence of a left adjoint as seen in the proof of Theorem 6.2. We repair this by working with weighted sets.

Definition 6.3.

1. A *weight function* on a set X is a function w from X into the nonnegative real numbers.

2. A *weighted set* is a pair (X, w) where X is a set and w is a weight function on X.

3. Let WSet be the category whose objects are weighted sets and whose morphisms are functions $f : (X_1, w_1) \to (X_2, w_2)$ satisfying $w_2(f(x)) \le w_1(x)$ for each $x \in X$.

Lemma 6.4. *There is a forgetful functor* $U : \boldsymbol{ba\ell} \to \mathsf{WSet}$*.*

Proof. If $A \in \boldsymbol{ba\ell}$, then $(A, \|\cdot\|) \in \mathsf{WSet}$. Moreover, if $\alpha : A \to B$ is a $\boldsymbol{ba\ell}$-morphism, then $\|\alpha(a)\| \le \|a\|$ by Lemma 6.1. Therefore, α is a WSet-morphism. Thus, the assignment $A \mapsto (A, \|\cdot\|)$ defines a forgetful functor $U : \boldsymbol{ba\ell} \to \mathsf{WSet}$. □

Definition 6.5. Let $A \in \boldsymbol{\ell alg}$. Call $a \in A$ *bounded* if there is $n \in \mathbb{N}$ with $-n \cdot 1 \le a \le n \cdot 1$. Let A^* be the set of bounded elements of A.

Let $A \in \boldsymbol{\ell alg}$. If $a, b \in A^*$, then there are $n, m \in \mathbb{N}$ with $-n \cdot 1 \le a \le n \cdot 1$ and $-m \cdot 1 \le b \le m \cdot 1$. Therefore, $-(n + m) \cdot 1 \le a \pm b \le (n + m) \cdot 1$. Similar facts hold for join, meet, and multiplication. Thus, we have the following:

Lemma 6.6. *Let* $A \in \boldsymbol{\ell alg}$*. Then* A^* *is a subalgebra of* A*, and hence* A^* *is a bounded* ℓ*-algebra. Therefore, if* A *is archimedean, then* $A^* \in \boldsymbol{ba\ell}$*.*

Let $A \in \boldsymbol{\ell alg}$. As we pointed out in Section 5.1, ℓ-ideals are kernels of ℓ-algebra homomorphisms. However, if I is an ℓ-ideal of A, then the quotient A/I may not be archimedean even if A is archimedean.

Definition 6.7. We call an ℓ-ideal I of $A \in \boldsymbol{\ell alg}$ *archimedean* if A/I is archimedean.

Remark 6.8. Archimedean ℓ-ideals were studied by Banaschewski (see [4, App. 2], [5]) in the category of archimedean f-rings.

It is easy to see that the intersection of archimedean ℓ-ideals is archimedean. Therefore, we may talk about the archimedean ℓ-ideal of A generated by $S \subseteq A$.

Theorem 6.9. *The forgetful functor* $U : \boldsymbol{ba\ell} \to \mathsf{WSet}$ *has a left adjoint.*

Proof. It is enough to show that there is a free object in $\boldsymbol{ba\ell}$ on each $(X, w) \in \mathsf{WSet}$ (see, e.g., [2, Ex. 18.2(2)]). Let $G(X)$ be the free object in $\boldsymbol{\ell alg}$ on X and let $g : X \to G(X)$ be the corresponding map. We next quotient $G(X)$ by an archimedean ℓ-ideal I so that $-w(x) \le g(x) + I \le w(x)$ for each $x \in X$. Let I be the archimedean ℓ-ideal of $G(X)$ generated by

$$\{g(x) - ((g(x) \vee -w(x)) \wedge w(x)) \mid x \in X\},$$

and set $F(X, w) = G(X)/I$. Let $\pi : G(X) \to F(X, w)$ be the canonical projection. Clearly $F(X, w)$ is an archimedean ℓ-algebra. We show that $F(X, w)$ is bounded, and hence that $F(X, w) \in \boldsymbol{ba\ell}$. Let $G(X)^*$ be the bounded subalgebra of $G(X)$ (see Lemma 6.6). Since $G(X)$ is generated by $\{g(x) \mid x \in X\}$, we have that $G(X)/I$ is generated by $\{\pi g(x) \mid x \in X\}$. Now,

$$\pi g(x) = \pi((g(x) \vee -w(x)) \wedge w(x))$$

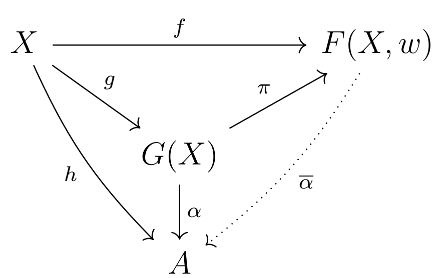
since $g(x) - ((g(x) \vee -w(x)) \wedge w(x)) \in I$. We have $-w(x) \le (g(x) \vee -w(x)) \wedge w(x) \le w(x)$, so $(g(x) \vee -w(x)) \wedge w(x) \in G(X)^*$. This shows that the generators of $F(X, w)$ lie in $\pi[G(X)^*]$, so $F(X, w) \cong G(X)^*/(I \cap G(X)^*)$ is a quotient of $G(X)^*$. Thus, $F(X, w)$ is bounded.

Let $f : X \to F(X, w)$ be given by $f(x) = \pi g(x)$. Since $f(x) = \pi((g(x) \vee -w(x)) \wedge w(x))$, we have $-w(x) \le f(x) \le w(x)$, so $\|f(x)\| \le w(x)$. Therefore, f is a WSet-morphism.

Let $A \in$ ***baℓ*** and $h : X \to A$ be a WSet-morphism, so $\|h(x)\| \leq w(x)$ for each $x \in X$. There is an ℓ-algebra homomorphism $\alpha : G(X) \to A$ with $\alpha \circ g = h$. Because A is archimedean, $G(X)/\ker(\alpha)$ is archimedean, so $\ker(\alpha)$ is an archimedean ℓ-ideal of $G(X)$. We show that $I \subseteq \ker(\alpha)$. It suffices to show that $g(x) - ((g(x) \vee -w(x)) \wedge w(x)) \in \ker(\alpha)$ for each $x \in X$ since $\ker(\alpha)$ is an archimedean ℓ-ideal. Because $\|h(x)\| \leq w(x)$, we have $-w(x) \leq h(x) \leq w(x)$. Therefore,

$$\begin{aligned}
\alpha((g(x) \vee -w(x)) \wedge w(x)) &= (\alpha g(x) \vee -w(x)) \wedge w(x) \\
&= (h(x) \vee -w(x)) \wedge w(x) \\
&= h(x) \\
&= \alpha g(x),
\end{aligned}$$

and hence $\alpha(g(x) - ((g(x) \vee -w(x)) \wedge w(x))) = 0$. Thus, $I \subseteq \ker(\alpha)$, so there is a well-defined ℓ-algebra homomorphism $\overline{\alpha} : F(X, w) \to A$ satisfying $\overline{\alpha} \circ \pi = \alpha$. Consequently, $\overline{\alpha} \circ f = \overline{\alpha} \circ \pi \circ g = \alpha \circ g = h$.



It is left to show uniqueness of $\overline{\alpha}$. Let $\gamma : F(X, w) \to A$ be a ***baℓ***-morphism satisfying $\gamma \circ f = h$. If $\alpha' = \gamma \circ \pi$, then $\alpha' : G(X) \to A$ is an ***ℓalg***-morphism and we have that $\alpha' \circ g = \gamma \circ \pi \circ g = \gamma \circ f = h$. Since $G(X)$ is a free object in ***ℓalg*** and $\alpha' \circ g = h = \alpha \circ g$, uniqueness implies that $\alpha' = \alpha$. From this we get $\gamma \circ \pi = \alpha = \overline{\alpha} \circ \pi$. Because π is onto, we conclude that $\gamma = \overline{\alpha}$. □

Remark 6.10. If $(X,w) \in \mathsf{WSet}$, then $\|f(x)\| = w(x)$. To see this, since $w : (X,w) \to (\mathbb{R}, |\cdot|)$ is a WSet-morphism, by Theorem 6.9, there is a $\boldsymbol{ba\ell}$-morphism $\alpha : F(X,w) \to \mathbb{R}$ with $\alpha \circ f = w$. Because f is a weighted set morphism, by Lemma 6.1 we have $w(x) = \|\alpha(f(x))\| \leq \|f(x)\| \leq w(x)$. Thus, $\|f(x)\| = w(x)$.

We next show that the Yosida space $Y_{F(X,w)}$ of $F(X,w)$ is homeomorphic to a power of $[0,1]$, and that $F(X,w)$ embeds into the ℓ-algebra of piecewise polynomial functions on $Y_{F(X,w)}$. For a set Z we let $PP([0,1]^Z)$ be the ℓ-algebra of piecewise polynomial functions on $[0,1]^Z$. If Z is finite, then the definition of $PP([0,1]^Z)$ is standard (see, e.g., [45, p. 651]). If Z is infinite, we define $PP([0,1]^Z)$ as the direct limit of $\{PP([0,1]^Y) \mid Y \text{ a finite subset of } Z\}$. It is straightforward to see that $PP([0,1]^Z) \in \boldsymbol{ba\ell}$.

Remark 6.11. For each $A \in \boldsymbol{ba\ell}$ and $M \in Y_A$ it is well known that $A/M \cong \mathbb{R}$ (see Remark 5.25). This allows us to identify the Yosida space Y_A with the space $\hom_{\boldsymbol{ba\ell}}(A, \mathbb{R})$ of $\boldsymbol{ba\ell}$-morphisms from A to $\mathbb{R}$, by sending $\alpha : A \to \mathbb{R}$ to $\ker(\alpha)$ and $M \in Y_A$ to the natural homomorphism $A \to \mathbb{R}$. The topology on $\hom_{\boldsymbol{ba\ell}}(A, \mathbb{R})$ is the subspace topology of the product topology on $\mathbb{R}^A$.

Theorem 6.12. *Let $(X,w) \in \mathsf{WSet}$ and let $X' = \{x \in X \mid w(x) > 0\}$.*

1. *The Yosida space of $F(X,w)$ is homeomorphic to $[0,1]^{X'}$.*

2. *$F(X,w)$ embeds into $PP([0,1]^{X'})$.*

Proof. (1). We identify $Y_{F(X,w)}$ with $\hom_{\boldsymbol{ba\ell}}(F(X,w), \mathbb{R})$ as in the paragraph before the theorem. From the universal mapping property, we see that there is a homeomorphism between $\hom_{\boldsymbol{ba\ell}}(F(X,w), \mathbb{R})$ and $\hom_{\mathsf{WSet}}((X,w), (\mathbb{R}, |\cdot|))$. If $g : X \to \mathbb{R}$ is a WSet-morphism, then $|g(x)| \leq w(x)$, so $-w(x) \leq g(x) \leq w(x)$. Therefore, $\hom_{\mathsf{WSet}}((X,w),(\mathbb{R},|\cdot|)) = \Pi_{x\in X}[-w(x),w(x)]$. If $x \in X'$, then $[-w(x),w(x)]$ is homeomorphic to $[0,1]$, and if $x \notin X'$, then $[-w(x),w(x)] = \{0\}$. Thus, $\Pi_{x\in X}[-w(x),w(x)]$ is homeomorphic to $[0,1]^{X'}$, and hence $Y_{F(X,w)}$ is homeomorphic to $[0,1]^{X'}$.

(2). Let $\varphi : Y_{F(X,w)} \to \Pi_{x\in X'}[-w(x),w(x)]$ be the homeomorphism from the proof of (1) and let $\tau_x : [0,1] \to [-w(x),w(x)]$ be the homeomorphism given by $\tau_x(a) = 2w(x)a - w(x)$. If τ is the product of the τ_x, then $\tau : [0,1]^{X'} \to \Pi_{x\in X'}[-w(x),w(x)]$ is a homeomorphism, and so $\rho := \tau^{-1} \circ \varphi$ is a homeomorphism from $Y_{F(X,w)}$ to $[0,1]^{X'}$. Therefore, $C(\rho) : C(Y_{F(X,w)}) \to C([0,1]^{X'})$ is a $\boldsymbol{ba\ell}$-isomorphism. Since $F(X,w)$ is generated by $f[X]$, it is sufficient to show that $C(\rho)(f(x)) \in PP([0,1]^{X'})$. Let $x \in X$. If $w(x) = 0$, then since $\|f(x)\| = w(x)$ (see Remark 6.10), $f(x) = 0$, so $C(\rho)(f(x)) = 0 \in PP([0,1]^{X'})$. Suppose that $w(x) > 0$. Then $C(\rho)(f(x)) = 2w(x)p_x - w(x) \in PP([0,1]^{X'})$, completing the proof. □

It is natural to ask whether free objects in $\boldsymbol{uba\ell}$ exist. The proof of Theorem 6.2 also yields that the forgetful functor $\boldsymbol{uba\ell} \to \mathsf{Sets}$ does not have a left adjoint. On the other hand, since the forgetful functor $\boldsymbol{ba\ell} \to \mathsf{WSet}$ has a left adjoint, if $\mathcal{C}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, then the forgetful functor $\mathcal{C} \to \mathsf{WSet}$ also has a left adjoint (because the composition of adjoints is an adjoint). Consequently, since $\boldsymbol{uba\ell}$ is a reflective subcategory of $\boldsymbol{ba\ell}$, we obtain:

Proposition 6.13. *The forgetful functor* $U : \boldsymbol{uba\ell} \to \mathsf{WSet}$ *has a left adjoint.*

Since taking uniform completion is the reflector $\boldsymbol{ba\ell} \to \boldsymbol{uba\ell}$, the left adjoint of Proposition 6.13 is obtained as the uniform completion of $F(X,w)$ for each $(X,w) \in \mathsf{WSet}$.

Remark 6.14. We finish this section by comparing our results with those in the vector lattice literature. Recall (see, e.g., [86, p. 48]) that the definition of a vector lattice, or Riesz space, is the same as that of an ℓ-algebra except that multiplication is not present in the signature, and so in vector lattices there is no analogue of the multiplicative identity.

1. Let VL be the category of vector lattices and vector lattice homomorphisms. Then VL is a variety, so free vector lattices exist by Birkhoff's theorem. Therefore, the forgetful functor $U : \mathsf{VL} \to \mathsf{Sets}$ has a left adjoint.

2. Let a *pointed vector lattice* be a vector lattice with a prescribed element, and a pointed vector lattice homomorphism a vector lattice homomorphism preserving the prescribed element. The associated category pVL is a variety, so the forgetful functor $U : \mathsf{pVL} \to \mathsf{Sets}$ has a left adjoint.

3. If we consider the full subcategory uVL of pVL consisting of pointed vector lattices whose prescribed element is a strong order-unit, then Birkhoff's theorem does not apply since uVL is not a variety. In fact, an argument similar to the proof of Theorem 6.2 shows that the forgetful functor $U : \mathsf{uVL} \to \mathsf{Sets}$ does not have a left adjoint. However, a small modification of the proof of Theorem 6.9 yields that the forgetful functor $U : \mathsf{uVL} \to \mathsf{WSet}$ does have a left adjoint.

4. Baker [3, Thm. 2.4] showed that the free vector lattice $F(X)$ on a set X embeds in the vector lattice $PL(\mathbb{R}^X)$ of piecewise linear functions on $\mathbb{R}^X$. In fact, Baker showed that $F(X)$ is isomorphic to the vector sublattice of $PL(\mathbb{R}^X)$ generated by the projection functions. Theorem 6.12(2) is an analogue of Baker's result since the proof shows that $F(X, w)$ is isomorphic to the subalgebra of $PP([0,1]^{X'})$ generated by the projection functions. Beynon [9, Thm. 1] showed that if X is finite, then $F(X) = PL(\mathbb{R}^X)$.

The analogue of Beynon's result for ℓ-algebras is related to the famous Pierce-Birkhoff conjecture [33, p. 68] (see also [89, 88]).

6.2 The endofunctor $\mathcal{H} : \boldsymbol{ba\ell} \to \boldsymbol{ba\ell}$

We are now ready to define the endofunctor $\mathcal{H}$ on $\boldsymbol{ba\ell}$. We define $\mathcal{H}(A)$ as a quotient of the free bounded archimedean ℓ-algebra $F(A, w_A)$. Although, as we pointed out in Section 6.1, the norm is a weight function on A, we will work with a different weight function on A. We use w_A instead of the norm in order for a modal operator to be a weighted set morphism (see Lemma 6.21).

Definition 6.15. Let $A \in \boldsymbol{ba\ell}$. Define w_A on A by $w_A(a) = \max\{\|a\|, 1\}$.

The next definition is motivated by the axioms defining a modal operator on $A \in \boldsymbol{ba\ell}$ listed in Definition 5.16.

Definition 6.16. Let $A \in \boldsymbol{ba\ell}$.

1. Let $F(A)$ be the free object in $\boldsymbol{ba\ell}$ on the weighted set (A, w_A), and let $f_A : A \to F(A)$ be the associated map. We let I_A be the archimedean ℓ-ideal of $F(A)$ generated by the following elements, where $a, b \in A$ and $r \in \mathbb{R}$:

   (a) $f_A(a \wedge b) - f_A(a) \wedge f_A(b)$;

   (b) $f_A(r) - r - (1 - r)f_A(0)$;

   (c) $f_A(a^+) - f_A(a)^+$;

   (d) $f_A(a + r) - f_A(a) - f_A(r) + f_A(0)$;

   (e) $f_A(ra) - f_A(r)f_A(a)$ if $0 \le r$.

2. Let $\mathcal{H}(A) = F(A)/I_A$ and $h_A : A \to \mathcal{H}(A)$ be the composition of f_A with the quotient map $\pi : F(A) \to \mathcal{H}(A)$.

3. For $a \in A$ let $\square_a = h_A(a)$.

Remark 6.17. The set $\{\square_a \mid a \in A\}$ generates $\mathcal{H}(A)$, and these generators satisfy the following relations that are the analogues of the axioms of a modal operator:

(F1) $\square_{a \wedge b} = \square_a \wedge \square_b$.

(F2) $\square_r = r + (1-r)\square_0$.

(F3) $\square_{a^+} = (\square_a)^+$.

(F4) $\square_{a+r} = \square_a + \square_r - \square_0$.

(F5) $\square_{ra} = \square_r \square_a$ if $0 \leq r$.

Theorem 6.18. *$\mathcal{H}$ is a covariant endofunctor on $\boldsymbol{ba\ell}$.*

Proof. Let $\alpha : A \to B$ be a $\boldsymbol{ba\ell}$-morphism. Then $\alpha : (A, w_A) \to (B, w_B)$ is a weighted set morphism since

$$w_B(\alpha(a)) = \max\{\|\alpha(a)\|, 1\} \leq \max\{\|a\|, 1\} = w_A(a)$$

for each $a \in A$. Therefore, there is a unique $\boldsymbol{ba\ell}$-morphism $\tau : F(A) \to F(B)$ making the following diagram commute.

$$\begin{array}{ccc} A & \xrightarrow{f_A} & F(A) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\tau} \\ B & \xrightarrow[f_B]{} & F(B) \end{array}$$

We show that $\tau(I_A) \subseteq I_B$. From this it will follow that there is an induced $\boldsymbol{ba\ell}$-morphism $\overline{\tau} : \mathcal{H}(A) \to \mathcal{H}(B)$ such that $\overline{\tau} \circ h_A = h_B \circ \alpha$. To see that $\tau(I_A) \subseteq I_B$, it suffices to show that the five sets of generators (a)–(e) of I_A are sent to I_B by τ. Since the arguments are similar, we only give the argument for the generators of type (a).

Let $a, b \in A$. Then

$$\begin{aligned}
\tau(f_A(a \wedge b) - f_A(a) \wedge f_A(b)) &= \tau f_A(a \wedge b) - (\tau f_A(a) \wedge \tau f_A(b)) \\
&= f_B\alpha(a \wedge b) - (f_B\alpha(a) \wedge f_B\alpha(b)) \\
&= f_B(\alpha(a) \wedge \alpha(b)) - (f_B\alpha(a) \wedge f_B\alpha(b)) \\
&\in I_B.
\end{aligned}$$

Therefore, τ induces a $\boldsymbol{ba\ell}$-morphism $\overline{\tau} : \mathcal{H}(A) \to \mathcal{H}(B)$. We set $\mathcal{H}(\alpha) = \overline{\tau}$. It follows that $\mathcal{H}(\alpha)$ is a unique $\boldsymbol{ba\ell}$-morphism that makes the following diagram commute.

$$\begin{array}{ccc}
A & \xrightarrow{\;h_A\;} & \mathcal{H}(A) \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mathcal{H}(\alpha)} \\
B & \xrightarrow[\;h_B\;]{} & \mathcal{H}(B)
\end{array}$$

It is clear that $\mathcal{H}$ sends identity morphisms to identity morphisms. If $\alpha : A \to B$ and $\gamma : B \to C$ are $\boldsymbol{ba\ell}$-morphisms, then

$$\mathcal{H}(\gamma \circ \alpha) \circ h_A = h_C \circ \gamma \circ \alpha = \mathcal{H}(\gamma) \circ h_B \circ \alpha = \mathcal{H}(\gamma) \circ \mathcal{H}(\alpha) \circ h_A.$$

Since $h_A[A]$ generates $\mathcal{H}(A)$, we see that $\mathcal{H}(\gamma \circ \alpha) = \mathcal{H}(\gamma) \circ \mathcal{H}(\alpha)$. Thus, $\mathcal{H}$ is a covariant functor. $\square$

Remark 6.19. From the commutativity $\mathcal{H}(\alpha) \circ h_A = h_B \circ \alpha$ it follows that $\mathcal{H}(\alpha)(\square_a) = \square_{\alpha(a)}$ for each $a \in A$. This will be used subsequently.

6.3 $\mathsf{Alg}(\mathcal{H})$ and $\boldsymbol{mba\ell}$

In this section we show that the category $\mathsf{Alg}(\mathcal{H})$ of algebras for the endofunctor $\mathcal{H}$ is isomorphic to $\boldsymbol{mba\ell}$. This is the direct analogue of what happens with modal algebras, see [84, Prop. 3.12]. We start by recalling the definition of algebras for an endofunctor (see, e.g., [2, Def. 5.37]).

Definition 6.20. Let C be a category and $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ an endofunctor on C.

1. An *algebra* for $\mathcal{T}$ is a pair (A, f) where A is an object of C and $f : \mathcal{T}(A) \to A$ is a C-morphism.

2. Let (A_1, f_1) and (A_2, f_2) be two algebras for $\mathcal{T}$. A *morphism* between (A_1, f_1) and (A_2, f_2) is a C-morphism $\alpha : A_1 \to A_2$ such that the following square is commutative.

$$\begin{array}{ccc} \mathcal{T}(A_1) & \xrightarrow{\mathcal{T}(\alpha)} & \mathcal{T}(A_2) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow[\alpha]{} & A_2 \end{array}$$

3. Let $\mathsf{Alg}(\mathcal{T})$ be the category whose objects are algebras for $\mathcal{T}$ and whose morphisms are morphisms of algebras.

Lemma 6.21. *If $(A, \Box) \in \boldsymbol{mba\ell}$, then $\Box : (A, w_A) \to (A, \|\cdot\|)$ is a weighted set morphism.*

Proof. Let $0 \leq r \in \mathbb{R}$. We first show that $\Box r \leq \max\{r, 1\}$. If $r \leq 1$, then $\Box r \leq \Box 1 = 1$ by Lemma 5.19. If $1 \leq r$, then $\Box r = r + (1 - r)\Box 0 \leq r$ since $0 \leq \Box 0$, again by Lemma 5.19. Therefore, $\Box r \leq \max\{r, 1\}$.

We next show that $-\Box r \leq \Box(-r)$. We have $\Box 0 = \Box(-r + r) = \Box(-r) + \Box r - \Box 0$, so $0 \leq 2\Box 0 = \Box(-r) + \Box r$. Thus, $-\Box r \leq \Box(-r)$.

To finish the proof, let $r = \|a\|$. Then $-r \le a \le r$, so $\Box(-r) \le \Box a \le \Box r$. We have $\Box r \le \max\{r, 1\}$ and $-\Box r \le \Box(-r)$. Therefore,

$$-\max\{\|a\|, 1\} = -\max\{r, 1\} \le -\Box r \le \Box(-r) \le \Box a \le \Box r \le \max\{r, 1\} = \max\{\|a\|, 1\},$$

which implies that $\|\Box a\| \le \max\{\|a\|, 1\} = w_A(a)$. Thus, $\Box : (A, w_A) \to (A, \|\cdot\|)$ is a weighted set morphism. $\Box$

Lemma 6.22. *There is a covariant functor $\mathcal{M} : \mathsf{Alg}(\mathcal{H}) \to \boldsymbol{mba\ell}$ sending (A, σ) to $(A, \Box_\sigma)$, where $\Box_\sigma a = \sigma(\Box_a)$ for each $a \in A$, and an $\mathsf{Alg}(\mathcal{H})$-morphism α to itself.*

Proof. Let $(A, \sigma) \in \mathsf{Alg}(\mathcal{H})$ and define $\Box_\sigma$ on A by $\Box_\sigma a = \sigma(\Box_a)$. It follows from Definition 5.16 and Remark 6.17 that $(A, \Box_\sigma) \in \boldsymbol{mba\ell}$. If $\alpha : (A, \sigma) \to (A', \sigma')$ is an $\mathsf{Alg}(\mathcal{H})$-morphism,

$$\begin{array}{ccc} \mathcal{H}(A) & \xrightarrow{\ \sigma\ } & A \\ {\scriptstyle \mathcal{H}(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{H}(A') & \xrightarrow[\ \sigma'\ ]{} & A' \end{array}$$

then

$$\alpha(\Box_\sigma a) = \alpha\sigma(\Box_a) = \sigma'\mathcal{H}(\alpha)(\Box_a) = \sigma'(\Box_{\alpha(a)}) = \Box_{\sigma'}\alpha(a),$$

where the second-to-last equality follows from Remark 6.19. Therefore, α is an $\boldsymbol{mba\ell}$-morphism. It is clear that $\mathcal{M}$ preserves identity morphisms and compositions. Thus, $\mathcal{M}$ is a covariant functor. $\Box$

Lemma 6.23. *There is a covariant functor $\mathcal{N} : \boldsymbol{mba\ell} \to \mathsf{Alg}(\mathcal{H})$ sending $(A, \Box)$ to $(A, \sigma_\Box)$, where $\sigma_\Box(\Box_a) = \Box a$ for each $a \in A$, and an $\boldsymbol{mba\ell}$-morphism α to itself.*

Proof. Since $\Box$ is a weighted set morphism by Lemma 6.21, there is a $\boldsymbol{ba\ell}$-morphism $\tau : F(A) \to A$ satisfying $\tau f_A(a) = \Box a$ by Theorem 6.9. It is clear from Definitions 5.16(1) and 6.16(1) that $I_A \subseteq \ker(\tau)$, so there is a $\boldsymbol{ba\ell}$-morphism $\sigma_\Box : \mathcal{H}(A) \to A$ satisfying $\sigma_\Box(\Box_a) = \Box a$. We set $\mathcal{N}(A, \Box) = (A, \sigma_\Box) \in \mathsf{Alg}(\mathcal{H})$. If $\alpha : (A, \Box) \to (A', \Box')$ is an $\boldsymbol{mba\ell}$-morphism, we show that α is an $\mathsf{Alg}(\mathcal{H})$-morphism. For this we show that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{H}(A) & \xrightarrow{\sigma_\Box} & A \\ {\scriptstyle \mathcal{H}(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{H}(A') & \xrightarrow[\sigma_{\Box'}]{} & A' \end{array}$$

By Remark 6.19, $\mathcal{H}(\alpha)(\Box_a) = \Box_{\alpha(a)}$. Therefore, because α preserves $\Box$, we have $\alpha\sigma_\Box(\Box_a) = \alpha(\Box a) = \Box\alpha(a)$ and $\sigma_{\Box'}\mathcal{H}(\alpha)(\Box_a) = \sigma_{\Box'}(\Box_{\alpha(a)}) = \Box\alpha(a)$. As $\{\Box_a \mid a \in A\}$ generates $\mathcal{H}(A)$, we see that $\alpha \circ \sigma_\Box = \sigma_{\Box'} \circ \mathcal{H}(\alpha)$, so α is an $\mathsf{Alg}(\mathcal{H})$-morphism. It is clear that $\mathcal{N}$ preserves identity morphisms and compositions. Thus, $\mathcal{N}$ is a covariant functor. $\Box$

Theorem 6.24. *The functors $\mathcal{M}$ and $\mathcal{N}$ yield an isomorphism of categories between $\mathsf{Alg}(\mathcal{H})$ and $\boldsymbol{mba\ell}$.*

Proof. Let $(A, \sigma) \in \mathsf{Alg}(\mathcal{H})$. Then $\mathcal{M}(A, \sigma) = (A, \Box_\sigma)$. Therefore, $\mathcal{N}\mathcal{M}(A, \sigma) = (A, \sigma_{\Box_\sigma})$ where $\sigma_{\Box_\sigma}(\Box_a) = \Box_\sigma a = \sigma(\Box_a)$. Thus, $\sigma_{\Box_\sigma} = \sigma$, and so $\mathcal{N}\mathcal{M} = 1_{\mathsf{Alg}(\mathcal{H})}$.

Next, let $(A, \Box) \in \boldsymbol{mba\ell}$. Then $\mathcal{N}(A, \Box) = (A, \sigma_\Box)$. Therefore, $\mathcal{M}\mathcal{N}(A, \Box) = (A, \Box_{\sigma_\Box})$. But $\Box_{\sigma_\Box} a = \sigma_\Box(\Box_a) = \Box a$ by the definition of $\sigma_\Box$, so $\Box_{\sigma_\Box} = \Box$. Thus, $\mathcal{M}\mathcal{N} = 1_{\boldsymbol{mba\ell}}$. Consequently, $\mathcal{M}$ and $\mathcal{N}$ yield an isomorphism between $\mathsf{Alg}(\mathcal{H})$ and $\boldsymbol{mba\ell}$. $\Box$

6.4 $\mathcal{H}$ and the Vietoris endofunctor

In this section we relate $\mathcal{H}$ to the Vietoris endofunctor $\mathcal{V} : \mathsf{KHaus} \to \mathsf{KHaus}$ by showing that the Yosida space $Y_{\mathcal{H}(A)}$ for $A \in \boldsymbol{ba\ell}$ is homeomorphic to $\mathcal{V}(Y_A)$.

Lemma 6.25. *Let $A \in \boldsymbol{ba\ell}$. Define $g_A : A \to C(\mathcal{V}Y_A)$ by*

$$g_A(a)(F) = \begin{cases} \inf \zeta_A(a)(F) & \text{if } F \neq \varnothing; \\ 1 & \text{if } F = \varnothing. \end{cases}$$

Then $g_A : (A, w_A) \to (C(\mathcal{V}Y_A), \|\cdot\|)$ is a well-defined weighted set morphism.

Proof. To simplify notation we write g for g_A. To see that g is well defined it is sufficient to show that $g(a)$ is continuous for each $a \in A$. Let $r, s \in \mathbb{R}$ with $r < s$. We show that

$$g(a)^{-1}(r,s) = \begin{cases} \Box_{\zeta_A(a)^{-1}(r,\infty)} \cap \Diamond_{\zeta_A(a)^{-1}(-\infty,s)} & \text{if } 1 \notin (r,s) \\ (\Box_{\zeta_A(a)^{-1}(r,\infty)} \cap \Diamond_{\zeta_A(a)^{-1}(-\infty,s)}) \cup \Box_\varnothing & \text{if } 1 \in (r,s). \end{cases}$$

Suppose that $1 \notin (r,s)$. Then $g(a)(F) \in (r,s)$ implies that $F \neq \varnothing$. Therefore, since F is compact and hence $\zeta_A(a)$ attains its infimum on F, we have

$$\begin{aligned} F \in g(a)^{-1}(r,s) \quad &\text{iff} \quad r < \inf \zeta_A(a)(F) < s \\ &\text{iff} \quad r < \min \zeta_A(a)(F) < s \\ &\text{iff} \quad F \in \Box_{\zeta_A(a)^{-1}(r,\infty)} \cap \Diamond_{\zeta_A(a)^{-1}(-\infty,s)}. \end{aligned}$$

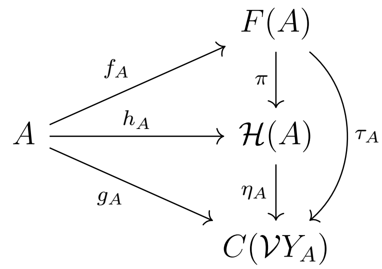
On the other hand, if $1 \in (r,s)$, then $\varnothing \in g(a)^{-1}(r,s)$. Therefore, since $\Box_\varnothing = \{\varnothing\}$, the calculation above yields the second case. Thus, $g(a)$ is continuous.

It is left to show that g is a weighted set morphism. Let $a \in A$. Then $w_A(a) = \max\{\|a\|, 1\}$. Suppose that $\|a\| = r$. Then $-r \leq a \leq r$. If F is nonempty, then $-r \leq \inf \zeta_A(a)(F) \leq r$, so $|\inf \zeta_A(a)(F)| \leq r$. Also, $g(a)(\varnothing) = 1$. Therefore,

$$\begin{aligned} \|g(a)\| &= \sup\{|g(a)(F)| \mid F \in \mathcal{V}(Y_A)\} = \sup\{\{|\inf \zeta_A(a)(F)| \mid F \neq \varnothing\} \cup \{1\}\} \\ &= \max\{\sup\{|\inf \zeta_A(a)(F)| \mid F \neq \varnothing\}, 1\} \leq \max\{r, 1\} = w_A(a). \end{aligned}$$

Thus, $g : (A, w_A) \to (C(\mathcal{V}Y_A), \|\cdot\|)$ is a weighted set morphism. $\square$

Lemma 6.26. *There is a (unique)* $\boldsymbol{bal}$*-morphism* $\tau_A : F(A) \to C(\mathcal{V}Y_A)$ *satisfying* $\tau_A \circ f_A = g_A$*, the image of* τ_A *is uniformly dense in* $C(\mathcal{V}Y_A)$*, and* $\ker(\tau_A)$ *contains* I_A*. Therefore, there is a (unique)* $\boldsymbol{bal}$*-morphism* $\eta_A : \mathcal{H}(A) \to C(\mathcal{V}Y_A)$ *satisfying* $\eta_A \circ h_A = g_A$ *and whose image is uniformly dense in* $C(\mathcal{V}Y_A)$*.*



Proof. The existence and uniqueness of τ_A follows from Lemma 6.25 and Theorem 6.9. To show that the image of τ_A is uniformly dense, by Lemma 5.5(2) it suffices to show that $\mathcal{Y}(\tau_A) : Y_{C(\mathcal{V}Y_A)} \to Y_{F(A)}$ is 1-1. We may identify $Y_{F(A)}$ with $\hom_{\boldsymbol{bal}}(F(A), \mathbb{R})$ by Remark 6.11 and $Y_{C(\mathcal{V}Y_A)}$ with $\mathcal{V}(Y_A)$ via the homeomorphism $\varepsilon_{\mathcal{V}Y_A}$ (see Section 5.1). Under these identifications, if $F \in \mathcal{V}Y_A$ we let $\rho_F \in \hom_{\boldsymbol{bal}}(F(A), \mathbb{R})$ be the corresponding homomorphism. For $a \in A$ and $r \in \mathbb{R}$ we have

$$\begin{aligned} \rho_F(f_A(a)) = r \quad &\text{iff} \quad f_A(a) - r \in \mathcal{Y}(\tau_A)(\varepsilon_{\mathcal{V}Y_A}(F)) \\ &\text{iff} \quad f_A(a) - r \in \tau_A^{-1}(\varepsilon_{\mathcal{V}Y_A}(F)) \\ &\text{iff} \quad \tau_A f_A(a) - r \in \varepsilon_{\mathcal{V}Y_A}(F) \\ &\text{iff} \quad \tau_A f_A(a)(F) = r \\ &\text{iff} \quad g_A(a)(F) = r. \end{aligned}$$

Therefore, ρ_F satisfies $\rho_F(f_A(a)) = \inf \zeta_A(a)(F)$ if $F \neq \varnothing$, and $\rho_\varnothing$ is the function sending each $f_A(a)$ to 1. To see that $\mathcal{Y}(\tau_A)$ is 1-1, suppose that $C \neq D$. If one of C, D is empty, say $C = \varnothing$, then $\rho_C f_A(0) = 1$ and $\rho_D f_A(0) = \inf \zeta_A(0)(D) = 0$ since D is nonempty. Therefore,

$\rho_C \neq \rho_D$. If $C, D \neq \varnothing$, without loss of generality we may assume that $C \nsubseteq D$. Then there is $y \in Y_A$ with $y \in C$ and $y \notin D$. Since Y_A is compact Hausdorff, there is $b \in C(Y_A)$ with $0 \leq b \leq 1$, $b(D) = \{1\}$ and $b(y) = 0$. Because $\zeta_A[A]$ is uniformly dense in $C(Y_A)$, there is $a \in A$ with $\|b - \zeta_A(a)\| < 1/3$. Therefore, $\inf \zeta_A(a)(D) \geq 2/3$ and $\inf \zeta_A(a)(C) \leq 1/3$. This shows that $\rho_C f_A(a) \neq \rho_D f_A(a)$, so $\rho_C \neq \rho_D$. Thus, $\mathcal{Y}(\tau_A)$ is 1-1, and hence the image of $\tau_A : F(A) \to C(\mathcal{V}Y_A)$ is uniformly dense.

To show that $I_A \subseteq \ker(\tau_A)$, it is sufficient to show that $\ker(\tau_A)$ contains all five classes of generators of I_A. Because the proof is similar to that of Lemma 5.14, we only demonstrate (a).

Let $a, b \in A$. We have

$$\tau_A(f_A(a \wedge b) - f_A(a) \wedge f_A(b)) = \tau_A f_A(a \wedge b) - (\tau_A f_A(a) \wedge \tau_A f_A(b)) = g_A(a \wedge b) - (g_A(a) \wedge g_A(b)).$$

Therefore, we need to prove that $g_A(a \wedge b) = g_A(a) \wedge g_A(b)$. Both sides send $\varnothing$ to 1. Suppose that $F \in \mathcal{V}(Y_A)$ is nonempty. Then

$$\begin{aligned} g_A(a \wedge b)(F) &= \inf(\zeta_A(a) \wedge \zeta_A(b))(F) = \min(\zeta_A(a) \wedge \zeta_A(b))(F) \\ &= \min\{(\zeta_A(a) \wedge \zeta_A(b))(x) \mid x \in F\} \\ &= \min\{\min\{\zeta_A(a)(x), \zeta_A(b)(x)\} \mid x \in F\} \\ &= \min\{\min \zeta_A(a)(F), \min \zeta_A(b)(F)\} \\ &= (g_A(a) \wedge g_A(b))(F). \end{aligned}$$

Thus, $g_A(a \wedge b) = g_A(a) \wedge g_A(b)$. □

We next show that η_A is 1-1. For this we require a technical result, which is an analogue of Proposition 5.30.

Definition 6.27. Let $A \in \boldsymbol{ba\ell}$.

1. If $x \in Y_{\mathcal{H}(A)}$, set $\Box^{-1}x = \{a \in A \mid \Box_a \in x\}$.

2. If $S \subseteq A$, set $S^+ = \{s \in S \mid 0 \leq s\}$.

3. Define a binary relation $R^\Box \subseteq Y_{\mathcal{H}(A)} \times Y_A$ by setting $xR^\Box y$ if $y^+ \subseteq \Box^{-1}x$ for each $x \in Y_{\mathcal{H}(A)}$ and $y \in Y_A$.

Proposition 6.28. *Let $A \in \boldsymbol{ba\ell}$ and $x \in Y_{\mathcal{H}(A)}$. Then $(\Box^{-1}x)^+ = \bigcup\{y^+ \mid y \in Y_A, xR^\Box y\}$.*

Proof. The proof is the same as that of Proposition 5.30 after replacing $\Box a$ with $\Box_a$ and $R_\Box$ with $R^\Box$. $\Box$

Lemma 6.29. *Let $\rho : \mathcal{H}(A) \to \mathbb{R}$ be a $\boldsymbol{ba\ell}$-morphism.*

1. *$\rho(\Box_0) \in \{0, 1\}$.*

2. *If $\rho(\Box_0) = 1$, then $\rho(\Box_a) = 1$ for each $a \in A$.*

Proof. (1) If we set $r = 0 = a$ in (F5) of Remark 6.17, we get $\Box_0\Box_0 = \Box_0$, so $\Box_0$ is an idempotent. Therefore, $\rho(\Box_0) \in \mathbb{R}$ is an idempotent, and hence $\rho(\Box_0) \in \{0, 1\}$.

(2) Suppose that $\rho(\Box_0) = 1$. By (F5), $\Box_0\Box_a = \Box_0$ for each $a \in A$. So applying ρ to both sides yields $\rho(\Box_a) = 1$. $\Box$

Theorem 6.30. *For $A \in \boldsymbol{ba\ell}$, the Yosida space of $\mathcal{H}(A)$ is homeomorphic to $\mathcal{V}(Y_A)$.*

Proof. The map $\eta_A : \mathcal{H}(A) \to C(\mathcal{V}Y_A)$ induces a continuous map $\mathcal{Y}(\eta_A) : Y_{C(\mathcal{V}Y_A)} \to Y_{\mathcal{H}(A)}$. We identify $Y_{C(\mathcal{V}Y_A)}$ with $\mathcal{V}(Y_A)$ and $Y_{\mathcal{H}(A)}$ with $\hom_{\boldsymbol{ba\ell}}(\mathcal{H}(A), \mathbb{R})$ as in Remark 6.11. As we saw in the proof of Lemma 6.26, under these identifications $\mathcal{Y}(\eta_A)(F) := \rho_F$ satisfies

$\rho_F(\Box_a) = \inf \zeta_A(a)(F)$ if F is nonempty, and $\rho_F(\Box_a) = 1$ if $F = \varnothing$. By Lemma 6.26, the image of η_A is uniformly dense in $C(\mathcal{V}Y_A)$. Therefore, $\mathcal{Y}(\eta_A)$ is 1-1 by Lemma 5.5(2).

To show that $\mathcal{Y}(\eta_A)$ is onto, let $\rho : \mathcal{H}(A) \to \mathbb{R}$ be a $\boldsymbol{ba\ell}$-morphism. If $\rho(\Box_0) = 1$, then $\rho(\Box_a) = 1$ for all $a \in A$ by Lemma 6.29(2). Therefore, ρ and $\rho_\varnothing$ agree on each $\Box_a$. Since these generate $\mathcal{H}(A)$, we see that $\rho = \rho_\varnothing$. By Lemma 6.29(1), we now may assume that $\rho(\Box_0) = 0$. By (F2), $\rho(\Box_r) = r$ for each $r \in \mathbb{R}$. Let

$$S = \{(a - \rho(\Box_a))^- \mid a \in A\}$$

and $F = \{M \in Y_A \mid S \subseteq M\}$, a closed subset of Y_A. We claim that $\rho = \rho_F$. Let $a \in A$ and $y \in F$. Then $(a - \rho(\Box_a))^- \in y$. This means $0 \leq (\zeta_A(a) - \rho(\Box_a))(y)$ by [26, Rem. 2.11], so $\rho(\Box_a) \leq \zeta_A(a)(y)$. Since this is true for all $y \in F$, we see that $\rho(\Box_a) \leq \inf \zeta_A(a)(F)$. Thus, it suffices to prove that for each $a \in A$ there is $y \in F$ with $\zeta_A(a)(y) = \rho(\Box_a)$. In other words, we need to show that there is $y \in F$ with $a - \rho(\Box_a) \in y$.

Let $x = \ker(\rho) \in Y_{\mathcal{H}(A)}$. If $a \in A$, then

$$\rho(\Box_{a-\rho(\Box_a)}) = \rho(\Box_a + \Box_{-\rho(\Box_a)} - \Box_0) = \rho(\Box_a) - \rho(\Box_a) = 0$$

by (F4) and the fact that $\rho(\Box_r) = r$. From this and (F3) we see that

$$\rho(\Box_{(a-\rho(\Box_a))^+}) = \rho(\Box^+_{a-\rho(\Box_a)}) = \rho(\Box_{a-\rho(\Box_a)})^+ = \max\{\rho(\Box_{a-\rho(\Box_a)}), 0\} = \max\{0, 0\} = 0,$$

which implies that $(a - \rho(\Box_a))^+ \in \Box^{-1}x$. By Proposition 6.28, there is $y \in Y_A$ with $xR^\Box y$ and $(a - \rho(\Box_a))^+ \in y$. We show that these two facts imply that $y \in F$ and $\rho(\Box_a) = \zeta_A(a)(y)$. Let $b \in A$. Since $A/y \cong \mathbb{R}$, there is $r \in \mathbb{R}$ with $b - r \in y$. Therefore, $(b - r)^+ \in y$, so $\Box_{(b-r)^+} \in x$. Because $x = \ker(\rho)$,

$$0 = \rho(\Box_{(b-r)^+}) = \rho(\Box^+_{b-r}) = \rho(\Box_{b-r})^+ = \max\{\rho(\Box_{b-r}), 0\} = \max\{\rho(\Box_b) - r, 0\},$$

so $\rho(\Box_b) \leq r$. Consequently, $b + y = r + y \geq \rho(\Box_b) + y$, and hence $b - \rho(\Box_b) + y \geq 0 + y$. This implies that $(b - \rho(\Box_b))^- \in y$. Since this is true for all $b \in A$, we get $S \subseteq y$, so $y \in F$. Moreover, for $b = a$ we have $(a - \rho(\Box_a))^+, (a - \rho(\Box_a))^- \in y$, so $a - \rho(\Box_a) \in y$. By the above, this shows that $\rho = \rho_F$, so $\mathcal{Y}(\eta_A)$ is onto. Thus, $\mathcal{Y}(\eta_A)$ is a homeomorphism. $\Box$

Remark 6.31. By Theorem 6.30, $Y_{\mathcal{H}(A)}$ is homeomorphic to $\mathcal{V}(Y_A)$. Under this homeomorphism, $R^\Box \subseteq Y_{\mathcal{H}(A)} \times Y_A$ is identified with the relation $R \subseteq \mathcal{V}(Y_A) \times Y_A$ given by FRy iff $y \in F$. From this it follows that $R[F] = F$, and for $U \subseteq Y_A$ open, we have $R^{-1}[U] = \Diamond_U$ and $R^{-1}[Y_A \setminus U] = \mathcal{V}(Y_A) \setminus \Box_U$. Consequently, R is a continuous relation, and hence so is $R^\Box$.

6.5 $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$

In this section we lift the dual adjunction between $\boldsymbol{ba\ell}$ and KHaus to a dual adjunction between $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$. We show that this dual adjunction restricts to a dual equivalence between the reflective subcategory $\mathsf{Alg}^u(\mathcal{H})$ of $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$. The category $\mathsf{Alg}^u(\mathcal{H})$ consists of those $(A, \alpha) \in \mathsf{Alg}(\mathcal{H})$ where $A \in \boldsymbol{uba\ell}$. This dual equivalence lifts Gelfand duality. We conclude the section by giving an alternate description of $\mathsf{Alg}^u(\mathcal{H})$ as $\mathsf{Alg}(\mathcal{H}^u)$ where $\mathcal{H}^u$ is the endofunctor $\mathcal{CYH} : \boldsymbol{uba\ell} \to \boldsymbol{uba\ell}$.

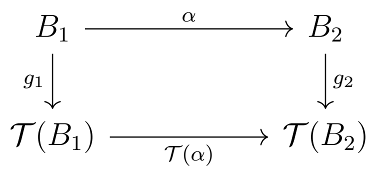
$$\boldsymbol{uba\ell} \xrightarrow{\mathcal{H}} \boldsymbol{ba\ell} \xrightarrow{\mathcal{Y}} \mathsf{KHaus} \xrightarrow{\mathcal{C}} \boldsymbol{uba\ell}$$

We start by recalling the definition of coalgebras (see, e.g., [112, Def. 9.1]), which is dual to the definition of algebras for an endofunctor.

Definition 6.32.

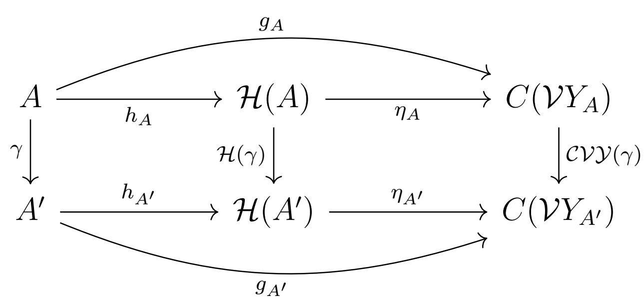
1. A *coalgebra* for an endofunctor $\mathcal{T} : \mathsf{C} \to \mathsf{C}$ is a pair (B, g) where B is an object of C and $g : B \to \mathcal{T}(B)$ is a C-morphism.

2. A *morphism* between two coalgebras (B_1, g_1) and (B_2, g_2) for $\mathcal{T}$ is a C-morphism $\alpha : B_1 \to B_2$ such that the following square is commutative.



3. Let $\mathsf{Coalg}(\mathcal{T})$ be the category whose objects are coalgebras for $\mathcal{T}$ and whose morphisms are morphisms of coalgebras.

Lemma 6.33. *Let $\gamma : A \to A'$ be a $\boldsymbol{ba\ell}$-morphism. Then the following diagram is commutative.*



Proof. By Remark 6.19, $\mathcal{H}(\gamma)(h_A(a)) = \mathcal{H}(\gamma)(\Box_a) = \Box_{\gamma(a)} = h_{A'}\gamma(a)$ for each $a \in A$. This shows that the left square of the diagram is commutative. By definition, $g_A = \eta_A \circ h_A$ and $g_{A'} = \eta_{A'} \circ h_{A'}$. We next show that the outside square is commutative, from which we then derive that the right square is commutative. Let $a \in A$ and $F \in \mathcal{V}(Y_{A'})$. If $F = \varnothing$, then

$$\mathcal{CVY}(\gamma)(g_A(a))(\varnothing) = g_A(a)(\mathcal{Y}(\gamma)(\varnothing)) = g_A(a)(\varnothing) = 1 = g_{A'}\gamma(a)(\varnothing).$$

If $F \neq \varnothing$, then naturality of ζ yields

$$\begin{aligned}\mathcal{CVY}(\gamma)(g_A(a))(F) &= g_A(a)(\mathcal{Y}(\gamma)(F)) = \inf(\zeta_A(a)\mathcal{Y}(\gamma))(F) \\ &= \inf(\mathcal{CY}(\gamma) \circ \zeta_A)(a)(F) = \inf \zeta_{A'}(\gamma(a))(F) \\ &= g_{A'}\gamma(a)(F).\end{aligned}$$

Thus, $\mathcal{CVY}(\gamma) \circ g_A = g_{A'} \circ \gamma$. Finally, to see that the right square is commutative,

$$\mathcal{CVY}(\gamma) \circ \eta_A \circ h_A = \mathcal{CVY}(\gamma) \circ g_A = g_{A'} \circ \gamma = \eta_{A'} \circ \mathcal{H}(\gamma) \circ h_A.$$

This yields $\mathcal{CVY}(\gamma) \circ \eta_A = \eta_{A'} \circ \mathcal{H}(\gamma)$ because the image of h_A generates $\mathcal{H}(A)$. $\square$

Proposition 6.34. *There is a contravariant functor* $\mathcal{F} : \mathsf{Alg}(\mathcal{H}) \to \mathsf{Coalg}(\mathcal{V})$.

Proof. By the proof of Theorem 6.30, if $A \in \boldsymbol{ba\ell}$, then $\mathcal{Y}(\eta_A)$ is a homeomorphism. For $(A, \alpha) \in \mathsf{Alg}(\mathcal{H})$, we set $\mathcal{F}(A, \alpha) = (Y_A, \mathcal{F}_\alpha) \in \mathsf{Coalg}(\mathcal{V})$, where

$$\mathcal{F}_\alpha = \varepsilon^{-1}_{\mathcal{V}(Y_A)} \circ \mathcal{Y}(\eta_A)^{-1} \circ \mathcal{Y}(\alpha) : Y_A \to \mathcal{V}(Y_A),$$

$$Y_A \xrightarrow{\mathcal{Y}(\alpha)} Y_{\mathcal{H}(A)} \xrightarrow{\mathcal{Y}(\eta_A)^{-1}} Y_{C(\mathcal{V}Y_A)} \xrightarrow{\varepsilon^{-1}_{\mathcal{V}(Y_A)}} \mathcal{V}(Y_A), \qquad Y_A \xrightarrow{\mathcal{F}_\alpha} \mathcal{V}(Y_A)$$

If $\gamma : (A, \alpha) \to (A', \alpha')$ is an $\mathsf{Alg}(\mathcal{H})$-morphism

$$\begin{array}{ccc} \mathcal{H}(A) & \xrightarrow{\alpha} & A \\ {\scriptstyle \mathcal{H}(\gamma)}\downarrow & & \downarrow{\scriptstyle \gamma} \\ \mathcal{H}(A') & \xrightarrow[\alpha']{} & A' \end{array}$$

then $\mathcal{Y}(\gamma) : Y_{A'} \to Y_A$ is a continuous map. We define $\mathcal{F}(\gamma) = \mathcal{Y}(\gamma)$. To see that $\mathcal{Y}(\gamma)$ is a $\mathsf{Coalg}(\mathcal{V})$-morphism, we show that the following diagram is commutative.

$$\begin{array}{ccc} Y_{A'} & \xrightarrow{\mathcal{F}_{\alpha'}} & \mathcal{V}(Y_{A'}) \\ {\scriptstyle \mathcal{Y}(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathcal{VY}(\gamma)} \\ Y_A & \xrightarrow[\mathcal{F}_\alpha]{} & \mathcal{V}(Y_A) \end{array} \tag{1}$$

To see this we first show that the following diagram is commutative.

$$\begin{array}{ccccc} \mathcal{V}(Y_{A'}) & \xrightarrow{\varepsilon_{\mathcal{V}(Y_{A'})}} & Y_{C(\mathcal{V}Y_{A'})} & \xrightarrow{\mathcal{Y}(\eta_{A'})} & Y_{\mathcal{H}(A')} \\ {\scriptstyle \mathcal{VY}(\gamma)}\downarrow & & \downarrow{\scriptstyle \mathcal{YCVY}(\gamma)} & & \downarrow{\scriptstyle \mathcal{YH}(\gamma)} \\ \mathcal{V}(Y_A) & \xrightarrow[\varepsilon_{\mathcal{V}(Y_A)}]{} & Y_{C(\mathcal{V}Y_A)} & \xrightarrow[\mathcal{Y}(\eta_A)]{} & Y_{\mathcal{H}(A)} \end{array} \tag{2}$$

The left square commutes due to the naturality of ε. For the right square, $\mathcal{Y}\mathcal{H}(\gamma) \circ \mathcal{Y}(\eta_{A'}) = \mathcal{Y}(\eta_{A'} \circ \mathcal{H}(\gamma))$ and $\mathcal{Y}(\eta_A) \circ \mathcal{Y}\mathcal{C}\mathcal{V}\mathcal{Y}(\gamma) = \mathcal{Y}(\mathcal{C}\mathcal{V}\mathcal{Y}(\gamma) \circ \eta_A)$. These are equal by Lemma 6.33.

Now, we show that Diagram (1) commutes. The equation

$$\mathcal{V}\mathcal{Y}(\gamma) \circ \mathcal{F}_{\alpha'} = \mathcal{F}_{\alpha} \circ \mathcal{Y}(\gamma)$$

is equivalent to

$$\mathcal{V}\mathcal{Y}(\gamma) \circ \varepsilon^{-1}_{\mathcal{V}(Y_{A'})} \circ \mathcal{Y}(\eta_{A'})^{-1} \circ \mathcal{Y}(\alpha') = \varepsilon^{-1}_{\mathcal{V}(Y_A)} \circ \mathcal{Y}(\eta_A)^{-1} \circ \mathcal{Y}(\alpha) \circ \mathcal{Y}(\gamma)$$

and therefore is equivalent to

$$\mathcal{Y}(\eta_A) \circ \varepsilon_{\mathcal{V}(Y_A)} \circ \mathcal{V}\mathcal{Y}(\gamma) \circ \varepsilon^{-1}_{\mathcal{V}(Y_{A'})} \circ \mathcal{Y}(\eta_{A'})^{-1} \circ \mathcal{Y}(\alpha') = \mathcal{Y}(\alpha) \circ \mathcal{Y}(\gamma). \tag{3}$$

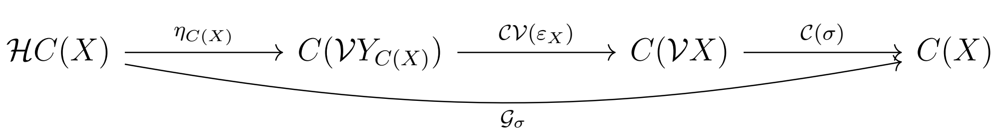
Using the commutativity of Diagram (2) and Equation (3), we see that commutativity of Diagram (1) is equivalent to the equation

$$\mathcal{Y}\mathcal{H}(\gamma) \circ \mathcal{Y}(\alpha') = \mathcal{Y}(\alpha) \circ \mathcal{Y}(\gamma).$$

Since γ is an $\mathsf{Alg}(\mathcal{H})$-morphism, we have $\gamma \circ \alpha = \alpha' \circ \mathcal{H}(\gamma)$. Applying $\mathcal{Y}$ to both sides then yields the commutativity of Diagram (1). Therefore, $\mathcal{Y}(\gamma)$ is a $\mathsf{Coalg}(\mathcal{V})$-morphism. It is then straightforward to see that $\mathcal{F}$ is a contravariant functor. □

Proposition 6.35. *There is a contravariant functor $\mathcal{G} : \mathsf{Coalg}(\mathcal{V}) \to \mathsf{Alg}(\mathcal{H})$.*

Proof. Let $(X, \sigma) \in \mathsf{Coalg}(\mathcal{V})$. Then $\mathcal{C}(\sigma) : C(\mathcal{V}X) \to C(X)$ is a $\boldsymbol{ba\ell}$-morphism. We set $\mathcal{G}(X, \sigma) = (X, \mathcal{G}_\sigma)$, where $\mathcal{G}_\sigma = \mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_X) \circ \eta_{C(X)}$.



If $\varphi : (X, \sigma) \to (X', \sigma')$ is a $\mathsf{Coalg}(\mathcal{V})$-morphism,

$$\begin{array}{ccc} X & \xrightarrow{\quad\sigma\quad} & \mathcal{V}(X) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{V}(\varphi)} \\ X' & \xrightarrow[\quad\sigma'\quad]{} & \mathcal{V}(X') \end{array}$$

We define $\mathcal{G}(\varphi) = \mathcal{C}(\varphi)$. We need to show that $\mathcal{C}(\varphi)$ is an $\mathsf{Alg}(\mathcal{H})$-morphism.

$$\begin{array}{ccc} \mathcal{H}C(X') & \xrightarrow{\quad\mathcal{G}_{\sigma'}\quad} & C(X') \\ {\scriptstyle\mathcal{H}\mathcal{C}(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{C}(\varphi)} \\ \mathcal{H}C(X) & \xrightarrow[\quad\mathcal{G}_{\sigma}\quad]{} & C(X) \end{array}$$

We have

$$\begin{aligned} \mathcal{C}(\varphi) \circ \mathcal{G}_{\sigma'} &= \mathcal{C}(\varphi) \circ \mathcal{C}(\sigma') \circ \mathcal{C}\mathcal{V}(\varepsilon_{X'}) \circ \eta_{C(X')} \\ &= \mathcal{C}(\sigma' \circ \varphi) \circ \mathcal{C}\mathcal{V}(\varepsilon_{X'}) \circ \eta_{C(X')} \\ &= \mathcal{C}(\mathcal{V}(\varphi) \circ \sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_{X'}) \circ \eta_{C(X')} \\ &= \mathcal{C}(\mathcal{V}(\varepsilon_{X'}) \circ \mathcal{V}(\varphi) \circ \sigma) \circ \eta_{C(X')} \\ &= \mathcal{C}(\mathcal{V}(\varepsilon_{X'} \circ \varphi) \circ \sigma) \circ \eta_{C(X')} \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{G}_\sigma \circ \mathcal{H}\mathcal{C}(\varphi) &= \mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_X) \circ \eta_{C(X)} \circ \mathcal{H}\mathcal{C}(\varphi) \\ &= \mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_X) \circ \mathcal{C}\mathcal{V}\mathcal{Y}\mathcal{C}(\varphi) \circ \eta_{C(X')} \\ &= \mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\mathcal{Y}\mathcal{C}(\varphi) \circ \varepsilon_X) \circ \eta_{C(X')} \\ &= \mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_{X'} \circ \varphi) \circ \eta_{C(X')} \\ &= \mathcal{C}(\mathcal{V}(\varepsilon_{X'} \circ \varphi) \circ \sigma) \circ \eta_{C(X')} \end{aligned}$$

where the second equality holds by applying Lemma 6.33 to $\gamma = \mathcal{C}(\varphi)$ and the fourth equality by the naturality of ε. Thus, $\mathcal{C}(\varphi) \circ \mathcal{G}_{\sigma'} = \mathcal{G}_\sigma \circ \mathcal{H}\mathcal{C}(\varphi)$. It is then straightforward to see that $\mathcal{G}$ is a contravariant functor. $\square$

Proposition 6.36. *There is a natural isomorphism* $\xi : 1_{\mathsf{Coalg}(\mathcal{V})} \to \mathcal{F}\mathcal{G}$.

Proof. We define $\xi : 1_{\mathsf{Coalg}(\mathcal{V})} \to \mathcal{F}\mathcal{G}$ as follows. If $(X, \sigma) \in \mathsf{Coalg}(\mathcal{V})$, then $\xi_{(X,\sigma)} = \varepsilon_X$.

$$\begin{array}{ccc} X & \xrightarrow{\quad\sigma\quad} & \mathcal{V}(X) \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \mathcal{V}(\varepsilon_X)} \\ Y_{C(X)} & \xrightarrow[\mathcal{F}_{\mathcal{G}_\sigma}]{\quad\quad} & \mathcal{V}(Y_{C(X)}) \end{array} \tag{4}$$

To see that ε_X is a $\mathsf{Coalg}(\mathcal{V})$-morphism, we have $\mathcal{G}_\sigma = \mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_X) \circ \eta_{C(X)}$. Therefore,

$$\begin{aligned} \mathcal{F}_{\mathcal{G}_\sigma} &= \varepsilon^{-1}_{\mathcal{V}(Y_{C(X)})} \circ \mathcal{Y}(\eta_{C(X)})^{-1} \circ \mathcal{Y}(\mathcal{G}_\sigma) \\ &= \varepsilon^{-1}_{\mathcal{V}(Y_{C(X)})} \circ \mathcal{Y}(\eta_{C(X)})^{-1} \circ \mathcal{Y}(\mathcal{C}(\sigma) \circ \mathcal{C}\mathcal{V}(\varepsilon_X) \circ \eta_{C(X)}) \\ &= \varepsilon^{-1}_{\mathcal{V}(Y_{C(X)})} \circ \mathcal{Y}(\eta_{C(X)})^{-1} \circ \mathcal{Y}(\eta_{C(X)}) \circ \mathcal{Y}\mathcal{C}\mathcal{V}(\varepsilon_X) \circ \mathcal{Y}\mathcal{C}(\sigma) \\ &= \varepsilon^{-1}_{\mathcal{V}(Y_{C(X)})} \circ \mathcal{Y}\mathcal{C}\mathcal{V}(\varepsilon_X) \circ \mathcal{Y}\mathcal{C}(\sigma) \\ &= \mathcal{V}(\varepsilon_X) \circ \varepsilon^{-1}_{\mathcal{V}(X)} \circ \mathcal{Y}\mathcal{C}(\sigma) \\ &= \mathcal{V}(\varepsilon_X) \circ \sigma \circ \varepsilon^{-1}_X \end{aligned}$$

where the last two equalities hold since ε is a natural isomorphism. Composing both sides on the right by ε_X shows that Diagram (4) commutes. Thus, ε_X is a $\mathsf{Coalg}(\mathcal{V})$-morphism.

To see that $\xi : 1_{\mathsf{Coalg}(\mathcal{V})} \to \mathcal{F}\mathcal{G}$ is a natural transformation, let $\varphi : (X, \sigma) \to (X', \sigma')$ be a $\mathsf{Coalg}(\mathcal{V})$-morphism. The following diagram commutes since ε is a natural transformation.

$$\begin{array}{ccc} X & \xrightarrow{\quad\varepsilon_X\quad} & Y_{C(X)} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}\mathcal{C}(\varphi)} \\ X' & \xrightarrow[\varepsilon_{X'}]{\quad\quad} & Y_{C(X')} \end{array}$$

Because $\xi_{(X,\sigma)} = \varepsilon_X$ and $\xi_{(X',\sigma')} = \varepsilon_{X'}$, it follows that ξ is natural. It is a natural isomorphism since $\xi_{(X,\sigma)} = \varepsilon_X$ is a homeomorphism for each $(X, \sigma) \in \mathsf{Coalg}(\mathcal{V})$. □

Remark 6.37. Since $\mathcal{C}$ and $\mathcal{Y}$ form a dual adjunction between $\boldsymbol{ba\ell}$ and KHaus, the natural transformations ζ and ε satisfy $\mathcal{Y}(\zeta_A) \circ \varepsilon_{Y_A} = 1_{Y_A}$ and $\mathcal{C}(\varepsilon_X) \circ \zeta_{C(X)} = 1_{C(X)}$ for each

$A \in \boldsymbol{ba\ell}$ and $X \in \mathsf{KHaus}$ by [87, Thm. IV.1.1]. Moreover, since ε is a natural isomorphism, $\mathcal{Y}(\zeta_A) = \varepsilon_{Y_A}^{-1}$ and $\zeta_{C(X)} = \mathcal{C}(\varepsilon_X)^{-1}$.

Proposition 6.38. *There is a natural transformation $\kappa : 1_{\mathsf{Alg}(\mathcal{H})} \to \mathcal{GF}$.*

Proof. We define $\kappa : 1_{\mathsf{Alg}(\mathcal{H})} \to \mathcal{GF}$ as follows. Let $(A, \alpha) \in \mathsf{Alg}(\mathcal{H})$. We set $\kappa_{(A,\alpha)} = \zeta_A$.

$$\begin{array}{ccc} \mathcal{H}(A) & \xrightarrow{\quad \alpha \quad} & A \\ {\scriptstyle \mathcal{H}(\zeta_A)}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_A} \\ \mathcal{H}C(Y_A) & \xrightarrow[\quad \mathcal{G}_{\mathcal{F}_\alpha} \quad]{} & C(Y_A) \end{array} \tag{5}$$

To see that ζ_A is an $\mathsf{Alg}(\mathcal{H})$-morphism, we show that Diagram (5) is commutative. We have $\mathcal{F}_\alpha = \varepsilon_{\mathcal{V}(Y_A)}^{-1} \circ \mathcal{Y}(\eta_A)^{-1} \circ \mathcal{Y}(\alpha)$ and so

$$\begin{aligned} \mathcal{G}_{\mathcal{F}_\alpha} &= \mathcal{C}(\mathcal{F}_\alpha) \circ \mathcal{CV}(\varepsilon_{Y_A}) \circ \eta_{C(Y_A)} \\ &= \mathcal{C}(\varepsilon_{\mathcal{V}(Y_A)}^{-1} \circ \mathcal{Y}(\eta_A)^{-1} \circ \mathcal{Y}(\alpha)) \circ \mathcal{CV}(\varepsilon_{Y_A}) \circ \eta_{C(Y_A)} \\ &= \mathcal{CY}(\alpha) \circ \mathcal{CY}(\eta_A)^{-1} \circ \mathcal{C}(\varepsilon_{\mathcal{V}(Y_A)})^{-1} \circ \mathcal{CV}(\varepsilon_{Y_A}) \circ \eta_{C(Y_A)} \\ &= \mathcal{CY}(\alpha) \circ \mathcal{CY}(\eta_A)^{-1} \circ \zeta_{C\mathcal{V}(Y_A)} \circ \mathcal{CV}(\varepsilon_{Y_A}) \circ \eta_{C(Y_A)} \\ &= \mathcal{CY}(\alpha) \circ \mathcal{CY}(\eta_A)^{-1} \circ \zeta_{C\mathcal{V}(Y_A)} \circ \mathcal{CVY}(\zeta_A)^{-1} \circ \eta_{C(Y_A)} \end{aligned}$$

because $\mathcal{C}(\varepsilon_{\mathcal{V}(Y_A)})^{-1} = \zeta_{C\mathcal{V}(Y_A)}$ and $\varepsilon_{Y_A} = \mathcal{Y}(\zeta_A)^{-1}$ by Remark 6.37. Thus, by Lemma 6.33 and the naturality of ζ (used twice),

$$\begin{aligned} \mathcal{G}_{\mathcal{F}_\alpha} \circ \mathcal{H}(\zeta_A) &= \mathcal{CY}(\alpha) \circ \mathcal{CY}(\eta_A)^{-1} \circ \zeta_{C\mathcal{V}(Y_A)} \circ \mathcal{CVY}(\zeta_A)^{-1} \circ \eta_{C(Y_A)} \circ \mathcal{H}(\zeta_A) \\ &= \mathcal{CY}(\alpha) \circ \mathcal{CY}(\eta_A)^{-1} \circ \zeta_{C\mathcal{V}(Y_A)} \circ \mathcal{CVY}(\zeta_A)^{-1} \circ \mathcal{CVY}(\zeta_A) \circ \eta_A \\ &= \mathcal{CY}(\alpha) \circ \mathcal{CY}(\eta_A)^{-1} \circ \zeta_{C\mathcal{V}(Y_A)} \circ \eta_A \\ &= \mathcal{CY}(\alpha) \circ \zeta_{\mathcal{H}(A)} \\ &= \zeta_A \circ \alpha. \end{aligned}$$

Thus, $\zeta_A \circ \alpha = \mathcal{G}_{\mathcal{F}_\alpha} \circ \mathcal{H}(\zeta_A)$, and hence ζ_A is a $\mathsf{Coalg}(\mathcal{V})$-morphism.

To show naturality, let $\gamma : (A, \alpha) \to (A', \alpha')$ be an $\mathsf{Alg}(\mathcal{H})$-morphism. The following diagram commutes since ζ is a natural transformation.

$$\begin{array}{ccc} A & \xrightarrow{\ \zeta_A\ } & C(Y_A) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle C\mathcal{Y}(\gamma)} \\ A' & \xrightarrow[\ \zeta_{A'}\ ]{} & C(Y_{A'}) \end{array}$$

Because $\kappa_{(A,\alpha)} = \zeta_A$ and $\kappa_{(A',\alpha')} = \zeta_{A'}$, it follows that κ is a natural transformation. □

Theorem 6.39. *The functors $\mathcal{F}$ and $\mathcal{G}$ yield a dual adjunction between $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$.*

Proof. By [87, Thm. IV.1.2] and Propositions 6.34–6.38, it suffices to show that

$$\mathcal{F}(\kappa_{(A,\alpha)}) \circ \xi_{\mathcal{F}(A,\alpha)} = 1_{\mathcal{F}(A,\alpha)}$$

and

$$\mathcal{G}(\xi_{(X,\sigma)}) \circ \kappa_{\mathcal{G}(X,\sigma)} = 1_{\mathcal{G}(X,\sigma)}$$

for each $(A, \alpha) \in \mathsf{Alg}(\mathcal{H})$ and $(X, \sigma) \in \mathsf{Coalg}(\mathcal{V})$. We have $\kappa_{(A,\alpha)} = \zeta_A$ and $\xi_{\mathcal{F}(A,\alpha)} = \varepsilon_{Y_A}$. Since $\mathcal{F}(\kappa_{(A,\alpha)}) = \mathcal{F}(\zeta_A) = \mathcal{Y}(\zeta_A)$ and $1_{\mathcal{F}(A,\alpha)} = 1_{Y_A}$, the first equation reduces to $\mathcal{Y}(\zeta_A) \circ \varepsilon_{Y_A} = 1_{Y_A}$, which holds by Remark 6.37. For the second equation, $\xi_{(X,\sigma)} = \varepsilon_X$ and $\kappa_{\mathcal{G}(X,\sigma)} = \zeta_{C(X)}$. Since $\mathcal{G}(\xi_{(X,\sigma)}) = \mathcal{G}(\varepsilon_X) = \mathcal{C}(\varepsilon_X)$ and $1_{\mathcal{G}(X,\sigma)} = 1_{C(X)}$, the equation $\mathcal{G}(\xi_{(X,\sigma)}) \circ \kappa_{\mathcal{G}(X,\sigma)} = 1_{\mathcal{G}(X,\sigma)}$ is equivalent to $\mathcal{C}(\varepsilon_X) \circ \zeta_{C(X)} = 1_{C(X)}$, which also holds by Remark 6.37. Therefore, $\mathcal{F}$ and $\mathcal{G}$ form a dual adjunction. □

Definition 6.40. Let $\mathsf{Alg}^u(\mathcal{H})$ be the full subcategory of $\mathsf{Alg}(\mathcal{H})$ consisting of those (A, α) with $A \in \boldsymbol{ubal}$.

Corollary 6.41.

1. *The functors* $\mathcal{F}$ *and* $\mathcal{G}$ *restrict to a dual equivalence between* $\mathsf{Alg}^u(\mathcal{H})$ *and* $\mathsf{Coalg}(\mathcal{V})$*.*

2. $\mathsf{Alg}^u(\mathcal{H})$ *is a reflective subcategory of* $\mathsf{Alg}(\mathcal{H})$*.*

Proof. (1) Let $(A,\alpha) \in \mathsf{Alg}(\mathcal{H})$. Then $\kappa_{(A,\alpha)} = \zeta_A$ is an isomorphism iff $A \in \boldsymbol{ubal}$ iff $(A,\alpha) \in \mathsf{Alg}^u(\mathcal{H})$. Consequently, $\kappa : 1_{\mathsf{Alg}^u(\mathcal{H})} \to \mathcal{GF}$ is a natural isomorphism by Proposition 6.38. Moreover, ξ is a natural isomorphism by Proposition 6.36. Therefore, $\mathcal{F}$ and $\mathcal{G}$ restrict to a dual equivalence between $\mathsf{Alg}^u(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$ by [87, Thm. IV.4.1].

(2) By (1), the functors $\mathcal{F}$ and $\mathcal{G}$ form a dual equivalence between $\mathsf{Alg}^u(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$. If $(A,\alpha) \in \mathsf{Alg}(\mathcal{H})$, then the morphism $\kappa_{(A,\alpha)}$ is a universal arrow from (A,α) to $\mathcal{F}$ by [87, Thm. IV.1.1]. Therefore, $\mathsf{Alg}^u(\mathcal{H})$ is a reflective subcatgory of $\mathsf{Alg}(\mathcal{H})$ (see [87, p. 89]). $\square$

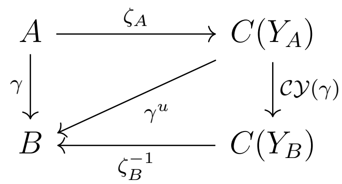
Proposition 6.42. *The functors* $\mathcal{M}, \mathcal{N}$ *yield an isomorphism between* $\mathsf{Alg}^u(\mathcal{H})$ *and* $\boldsymbol{mubal}$*.*

Proof. If $(A,\sigma) \in \mathsf{Alg}^u(\mathcal{H})$, then $A \in \boldsymbol{ubal}$, so $\mathcal{M}(A,\sigma) = (A, \Box_\sigma) \in \boldsymbol{mubal}$. If $(A,\Box) \in \boldsymbol{mubal}$, then $A \in \boldsymbol{ubal}$, so $\mathcal{N}(A,\Box) = (A, \sigma_\Box) \in \mathsf{Alg}^u(\mathcal{H})$. Therefore, the proof of Theorem 6.24 shows that $\mathcal{M}$ and $\mathcal{N}$ restrict to $\mathsf{Alg}^u(\mathcal{H})$ and $\boldsymbol{mubal}$, respectively, to yield an isomorphism. $\square$

We finish this section by giving an alternate view of the category $\mathsf{Alg}^u(\mathcal{H})$.

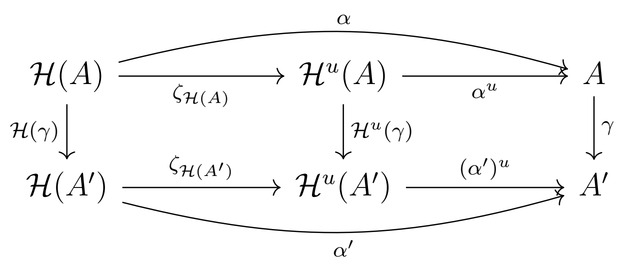
Definition 6.43. We let $\mathcal{H}^u$ be the endofunctor $\mathcal{CYH}$ on $\boldsymbol{ubal}$. Therefore, if $A \in \boldsymbol{ubal}$, then $\mathcal{H}^u(A) = C(Y_{\mathcal{H}(A)})$ and if $\alpha : A \to A'$ is a $\boldsymbol{ubal}$-morphism, then $\mathcal{H}^u(\alpha) = \mathcal{CYH}(\alpha)$.

By Proposition 5.4(2), if $\gamma : A \to B$ is a $\boldsymbol{bal}$-morphism with $B \in \boldsymbol{ubal}$, then there is a unique $\boldsymbol{ba\ell}$-morphism $\gamma^u : C(Y_A) \to B$ with $\gamma^u \circ \zeta_A = \gamma$, where $\gamma^u = \zeta_B^{-1} \circ \mathcal{CY}(\gamma)$.



Proposition 6.44. *There is an isomorphism of categories between* $\mathsf{Alg}^u(\mathcal{H})$ *and* $\mathsf{Alg}(\mathcal{H}^u)$.

Proof. We define $\mathcal{A} : \mathsf{Alg}^u(\mathcal{H}) \to \mathsf{Alg}(\mathcal{H}^u)$ on objects by sending (A, α) to (A, α^u). On morphisms, if γ is an $\mathsf{Alg}(\mathcal{H})$-morphism, then $\mathcal{A}(\gamma) = \gamma$.



To see that γ is an $\mathsf{Alg}(\mathcal{H}^u)$-morphism, the left square of the diagram commutes by the naturality of ζ. We have

$$(\gamma \circ \alpha^u) \circ \zeta_{\mathcal{H}(A)} = \gamma \circ \alpha = \alpha' \circ \mathcal{H}(\gamma) = (\alpha')^u \circ \zeta_{\mathcal{H}(A')} \circ \mathcal{H}(\gamma) = (\alpha')^u \circ \mathcal{H}^u(\gamma) \circ \zeta_{\mathcal{H}(A)}$$

so $\gamma \circ \alpha^u = (\alpha')^u \circ \mathcal{H}^u(\gamma)$ since $\zeta_{\mathcal{H}(A)}$ is epic. This shows that γ is an $\mathsf{Alg}(\mathcal{H}^u)$-morphism. It then follows that $\mathcal{A}$ is a covariant functor.

Going in the opposite direction, we define a functor $\mathcal{B} : \mathsf{Alg}(\mathcal{H}^u) \to \mathsf{Alg}^u(\mathcal{H})$ on objects by sending (A, α) to $(A, \alpha \circ \zeta_{\mathcal{H}(A)})$. On morphisms we send a $\mathsf{Alg}(\mathcal{H}^u)$-morphism $\gamma : A \to A'$ to itself. It is clear that $\mathcal{B}$ is a covariant functor.

If $(A, \alpha) \in \mathsf{Alg}^u(\mathcal{H})$, then $\mathcal{A}(A, \alpha) = (A, \alpha^u)$, and so $\mathcal{BA}(A, \alpha) = (A, \alpha^u \circ \zeta_{\mathcal{H}(A)}) = (A, \alpha)$. Therefore, $\mathcal{BA} = 1_{\mathsf{Alg}^u(\mathcal{H})}$. If $(A, \alpha) \in \mathsf{Alg}(\mathcal{H}^u)$, then $(A, \alpha \circ \zeta_{\mathcal{H}(A)}) \in \mathsf{Alg}^u(\mathcal{H})$, and $(\alpha \circ \zeta_{\mathcal{H}(A)})^u = \alpha$. Therefore, $\mathcal{AB} = 1_{\mathsf{Alg}(\mathcal{H}^u)}$. Thus, $\mathcal{A}, \mathcal{B}$ yield an isomorphism of categories between $\mathsf{Alg}^u(\mathcal{H})$ and $\mathsf{Alg}(\mathcal{H}^u)$. □

6.6 $\boldsymbol{mba\ell}$ and KHF

In this section we show how to derive from our results the dual adjunction between $\boldsymbol{mba\ell}$ and KHF and the dual equivalence between $\boldsymbol{muba\ell}$ and KHF obtained in Section 5.

We start by recalling (see, e.g., [15, Thm. 2.16]) that there is an isomorphism of categories between $\mathsf{Coalg}(\mathcal{V})$ and KHF. The isomorphism is determined by the following functors. The functor $\mathcal{S} : \mathsf{Coalg}(\mathcal{V}) \to \mathsf{KHF}$ sends (X, σ) to $(X, R_\sigma) \in \mathsf{KHF}$, where $xR_\sigma y$ if $y \in \sigma(x)$, and $\mathcal{S}$ sends a $\mathsf{Coalg}(\mathcal{V})$ morphism to itself. The functor $\mathcal{T} : \mathsf{KHF} \to \mathsf{Coalg}(\mathcal{V})$ sends $(X, R) \in \mathsf{KHF}$ to (X, σ_R), defined by $\sigma_R(x) = R[x]$, and sends a KHF-morphism to itself.

As a consequence of this and the results of the previous section, we obtain an alternate proof of Theorem 5.43.

Theorem 6.45. *There is a dual adjunction between* $\boldsymbol{mba\ell}$ *and* KHF *which restricts to a dual equivalence between* $\boldsymbol{muba\ell}$ *and* KHF*.*

Proof. By Theorem 6.39 the functors $\mathcal{F}$ and $\mathcal{G}$ form a dual adjunction between $\mathsf{Alg}(\mathcal{H})$ and $\mathsf{Coalg}(\mathcal{V})$. By Theorem 6.24, the functors $\mathcal{M}, \mathcal{N}$ yield an isomorphism of categories between $\mathsf{Alg}(\mathcal{H})$ and $\boldsymbol{mba\ell}$. The functors $\mathcal{S}, \mathcal{T}$ yield an isomorphism of categories between $\mathsf{Coalg}(\mathcal{V})$ and KHF [15, Thm. 2.16]. We thus have the following diagram.

$$\boldsymbol{muba\ell} \hookrightarrow \boldsymbol{mba\ell} \underset{\mathcal{M}}{\overset{\mathcal{N}}{\rightleftarrows}} \mathsf{Alg}(\mathcal{H}) \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathsf{Coalg}(\mathcal{V}) \underset{\mathcal{T}}{\overset{\mathcal{S}}{\rightleftarrows}} \mathsf{KHF}$$

Consequently, $\mathcal{SFN} : \boldsymbol{mba\ell} \to \mathsf{KHF}$ and $\mathcal{MGT} : \mathsf{KHF} \to \boldsymbol{mba\ell}$ yield a dual adjunction which restricts to a dual equivalence between $\boldsymbol{muba\ell}$ and KHF. ☐

Proposition 6.46. $\mathcal{SFN}$ *and* $\mathcal{MGT}$ *are precisely the functors* $\mathcal{C}$ *and* $\mathcal{Y}$ *yielding the dual adjunction of Theorem 5.43.*

Proof. Let $(A, \Box) \in \boldsymbol{mba\ell}$. Then $\mathcal{Y}(A, \Box) = (Y_A, R_\Box)$, where we recall from Definition 5.23 that $R_\Box$ is defined by $xR_\Box y$ if $y^+ \subseteq \Box^{-1}x$. We have $\mathcal{N}(A, \Box) = (A, \sigma_\Box)$, which satisfies $\sigma_\Box(\Box_a) = \Box a$ for all $a \in A$. Then $\mathcal{F}(A, \sigma_\Box) = (Y_A, \mathcal{F}_{\sigma_\Box})$, where we recall that $\mathcal{F}_{\sigma_\Box}$ is the composition $\varepsilon^{-1}_{\mathcal{V}(Y_A)} \circ \mathcal{Y}(\eta_A)^{-1} \circ \mathcal{Y}(\sigma_\Box)$. Finally, $\mathcal{S}$ sends this to $(Y_A, R_{\mathcal{F}_{\sigma_\Box}})$, where $xR_{\mathcal{F}_{\sigma_\Box}}y$ if $y \in \mathcal{F}_{\sigma_\Box}(x)$. Let $x \in Y_A$ and $F = \mathcal{F}_{\sigma_\Box}(x) \in \mathcal{V}(Y_A)$. If $M = \varepsilon_{\mathcal{V}(Y_A)}(F) \in Y_{C(\mathcal{V}Y_A)}$, then $M = \{g \in C(\mathcal{V}Y_A) \mid g(F) = 0\}$ and

$$\mathcal{Y}(\eta_A)(M) = \eta_A^{-1}(M) = \sigma_\Box^{-1}(x) = \mathcal{Y}(\sigma_\Box)(x).$$

We show that $R_\Box = R_{\mathcal{F}_{\sigma_\Box}}$. Suppose that $xR_\Box y$, so $\Box y^+ \subseteq x$. To see that $xR_{\mathcal{F}_{\sigma_\Box}}y$, we need to show that $y \in F$. If not, then by Urysohn's lemma and the fact that $\zeta_A[A]$ is uniformly dense in $C(Y_A)$, there is $a \in A$ with $\zeta_A(a)(y) = 0$ and $\zeta_A(a)[F] \geq 1/2$. By replacing a by a^+ we may assume that $a \geq 0$. Since $\zeta_A(a)(y) = 0$, we have $a \in y$. Therefore, $\Box a \in x$. This means $\sigma_\Box(\Box_a) \in x$, so $\Box_a \in \sigma_\Box^{-1}(x) = \eta_A^{-1}(M)$. Thus, $\eta_A(\Box_a) \in M$, so $g_A(a) \in M$. Therefore, $\inf g_A(\zeta_A(a))(F) = 0$, which is false by construction of a. This shows $y \in F$.

Conversely, if $xR_{\mathcal{F}_{\sigma_\Box}}y$, then $y \in F$. Let $a \in y^+$. Then $\inf g_A(\zeta_A(a))(F) = 0$ because $a \in y$ and $a \geq 0$. Therefore, $\eta_A(\Box_a) \in M$, so $\Box_a \in \eta_A^{-1}(M) = \sigma_\Box^{-1}(x)$. Thus, $\Box a = \sigma_\Box(\Box_a) \in x$. This shows $\Box y^+ \subseteq x$, so $xR_\Box y$, completing the proof that $R_{\mathcal{F}_{\sigma_\Box}} = R_\Box$. Therefore, $\mathcal{Y}$ and $\mathcal{SFN}$ agree on the objects of $\boldsymbol{mba\ell}$. For morphisms, if $\alpha : (A, \Box) \to (A', \Box')$ is an $\boldsymbol{mba\ell}$-morphism, then $\mathcal{SFN}(\alpha) = \mathcal{SF}(\alpha) = \mathcal{S}(\mathcal{Y}(\alpha)) = \mathcal{Y}(\alpha)$. Thus, $\mathcal{SFN} = \mathcal{Y}$.

In the opposite direction, if $(X, R) \in \mathsf{KHF}$, we show that $\mathcal{C}(X, R) = \mathcal{MGT}(X, R)$. First, $\mathcal{C}(X, R) = (C(X), \Box_R)$, where we recall from Section 5.2 that $\Box_R f$ is given by

$$(\Box_R f)(x) = \begin{cases} \inf fR[x] & \text{if } R[x] \neq \varnothing \\ 1 & \text{if } R[x] = \varnothing. \end{cases}$$

The functor $\mathcal{T}$ sends (X,R) to (X,σ_R), where $\sigma_R(x) = R[x]$. Then $\mathcal{G}$ sends this to $(C(X),\mathcal{G}_{\sigma_R})$, where we recall that $\mathcal{G}_{\sigma_R} = \mathcal{C}(\sigma_R)\circ\mathcal{CV}(\varepsilon_X)\circ\eta_{C(X)}$. Finally, $(C(X),\mathcal{G}_{\sigma_R})$ is sent by $\mathcal{M}$ to $(C(X),\Box_{\mathcal{G}_{\sigma_R}})$, where $\Box_{\mathcal{G}_{\sigma_R}}f = \mathcal{G}_{\sigma_R}(\Box_f)$. We have

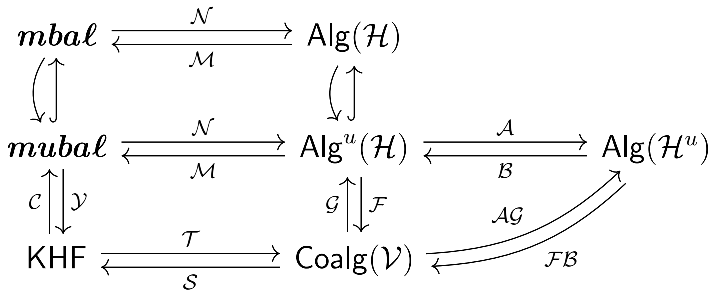
$$\begin{aligned}\mathcal{G}_{\sigma_R}(\Box_f) &= \mathcal{C}(\sigma_R)(\mathcal{CV}(\varepsilon_X)(\eta_{C(X)}(\Box_f)))\\ &= \mathcal{C}(\sigma_R)(\mathcal{CV}(\varepsilon_X)(g_{C(X)}(f)))\\ &= \mathcal{C}(\sigma_R)(g_{C(X)}(f)\circ\mathcal{V}(\varepsilon_X))\\ &= g_{C(X)}(f)\circ\mathcal{V}(\varepsilon_X)\circ\sigma_R.\end{aligned}$$

Let $x\in X$. Then $\sigma_R(x) = R[x]$ and $\mathcal{V}(\varepsilon_X)(R[x]) = \varepsilon_X(R[x])$. Therefore, since $f = \zeta_{C(X)}(f)\circ\varepsilon_X$ by Remark 6.37, we have

$$\begin{aligned}g_{C(X)}(f)(\varepsilon_X R[x]) &= \begin{cases}\inf\zeta_{C(X)}(f)(\varepsilon_X R[x]) & \text{if } R[x]\neq\varnothing\\ 1 & \text{if } R[x]=\varnothing\end{cases}\\ &= \begin{cases}\inf fR[x] & \text{if } R[x]\neq\varnothing\\ 1 & \text{if } R[x]=\varnothing\end{cases}\\ &= (\Box_R f)(x).\end{aligned}$$

Thus, $\mathcal{C}$ and $\mathcal{MGT}$ agree on objects of KHF. If $\sigma:(X,R)\to(X',R')$ is a KHF-morphism, then $\mathcal{MGT}(\sigma)=\mathcal{MC}(\sigma)=\mathcal{C}(\sigma)$. Consequently, $\mathcal{MGT}=\mathcal{C}$. $\Box$

The following diagram shows the relationship between the various categories we have considered in Part II, where the curved vertical arrows are reflections and the vertical hook-arrows are full embeddings.



Remark 6.47. The Vietoris space of X is usually defined as the space of nonempty closed subsets of X (see, e.g., [47, p. 120]). However, we follow [76, p. 111] in including $\varnothing$ in $\mathcal{V}(X)$. This is necessary for our considerations since the continuous relation R on X may not be serial, and hence there may be $x \in X$ with $R[x] = \varnothing$. Therefore, $\rho_R(x) = \varnothing$, and we need $\varnothing \in \mathcal{V}(X)$ for ρ_R to be well defined. It is straightforward to see that the category of compact Hausdorff frames with a serial relation is isomorphic to the category $\mathsf{Coalg}(\mathcal{V}^*)$ where $\mathcal{V}^*$ is the endofunctor on KHaus defined by $\mathcal{V}^*(X) = \mathcal{V}(X) \setminus \{\varnothing\}$. In [20, Sec. 7] we prove there is a dual adjunction between the category of compact Hausdorff frames with a serial relation and the subcategory of $\boldsymbol{mba\ell}$ given by the algebras satisfying $\Box 0 = 0$ that restricts to a dual equivalence on the subcategory of the uniformly complete algebras. Such a result can be obtained via an algebraic/coalgebraic approach analogous to the one presented in this section. Indeed, in [21, Sec. 8] we show how to simplify the construction of $\mathcal{H}$ to obtain an endofunctor $\mathcal{H}^*$ on $\boldsymbol{ba\ell}$ dual to $\mathcal{V}^*$ and we prove that there is a dual adjunction between $\mathsf{Coalg}(\mathcal{V}^*)$ and $\mathsf{Alg}(\mathcal{H}^*)$ that restricts to a dual equivalence.

6.7 Connection to modal algebras and descriptive frames

In this section we connect our results with those of Abramsky [1] and Kupke, Kurz, and Venema [84].

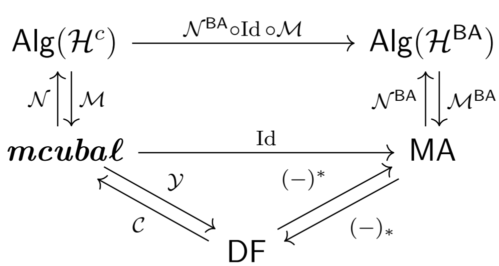
Lemma 6.48. *If* $A \in \boldsymbol{cuba\ell}$*, then* $\mathcal{H}^u(A) \in \boldsymbol{cuba\ell}$*.*

Proof. By [24, Prop. 5.20], if $A \in \boldsymbol{cuba\ell}$, then Y_A is a Stone space. Therefore, $\mathcal{V}(Y_A)$ is a Stone space, and hence $Y_{\mathcal{H}^u(A)}$ is a Stone space by Theorem 6.30. Thus, $\mathcal{H}^u(A) \in \boldsymbol{cuba\ell}$ by [24, Prop. 5.20]. $\square$

To distinguish between $\mathcal{V}$ on KHaus and Stone, we denote the Vietoris endofunctor on Stone by $\mathcal{V}^{\mathsf{S}}$. By Lemma 6.48, $\mathcal{H}^u$ restricts to an endofunctor on $\boldsymbol{cuba\ell}$, which we denote by $\mathcal{H}^c$. The following result is then an immediate consequence of Corollary 6.41(1).

Theorem 6.49. *There is a dual equivalence between* $\mathsf{Alg}^u(\mathcal{H}^c)$ *and* $\mathsf{Coalg}(\mathcal{V}^{\mathsf{S}})$*.*

We let $\mathcal{H}^{\mathsf{BA}}$ be the functor of [84] that sends $B \in \mathsf{BA}$ to the free boolean algebra over its underlying meet-semilattice. It was shown in [84, Prop., 3.12] that $\mathsf{Alg}(\mathcal{H}^{\mathsf{BA}})$ is isomorphic to the category MA of modal algebras. In parallel of $\mathcal{M} : \mathsf{Alg}(\mathcal{H}) \to \boldsymbol{mba\ell}$ and $\mathcal{N} : \boldsymbol{mba\ell} \to \mathsf{Alg}(\mathcal{H})$, we denote the functors giving the isomorphism by $\mathcal{M}^{\mathsf{BA}} : \mathsf{Alg}(\mathcal{H}^{\mathsf{BA}}) \to \mathsf{MA}$ and $\mathcal{N}^{\mathsf{BA}} : \mathsf{MA} \to \mathsf{Alg}(\mathcal{H}^{\mathsf{BA}})$. By Theorem 5.54, the triangle in the diagram below commutes up to natural isomorphism, where $(-)^* : \mathsf{DF} \to \mathsf{MA}$ and $(-)_* : \mathsf{MA} \to \mathsf{DF}$ are the functors yielding Jónsson-Tarski duality, and the functor Id sends $(A, \Box) \in \boldsymbol{mba\ell}$ to $(\mathrm{Id}(A), \Box|_{\mathrm{Id}(A)})$ (see Lemma 5.50). Therefore, there is an equivalence of categories between $\mathsf{Alg}(\mathcal{H}^c)$ and $\mathsf{Alg}(\mathcal{H}^{\mathsf{BA}})$, where the functor $\mathsf{Alg}(\mathcal{H}^c) \to \mathsf{Alg}(\mathcal{H}^{\mathsf{BA}})$ is the composition $\mathcal{N}^{\mathsf{BA}} \circ \mathrm{Id} \circ \mathcal{M}$.



The diagram displays the category DF at the bottom and four different categories dually equivalent to it. Thus, it shows various ways to obtain Jónsson-Tarski duality and connects them via the horizontal and vertical functors. The right-hand side contains the classical version of Jónsson-Tarski duality and the algebra/coalgebra approach of [84]. The left-hand side presents two new ways to obtain Jónsson-Tarski duality described in Section 5.5 and in this section.

6.8 Open problems and future directions of research

We conclude by listing several open problems and possible future directions of research pertaining to the second part of the thesis.

(1) As we pointed out in the Introduction, there are other dualities for KHaus. For example, in pointfree topology we have Isbell duality [75] (see also [6] or [76, Sec. III.1]) and de Vries duality [44] (see also [13]). The two are closely related, see [14]. Isbell and de Vries dualities were generalized to the setting of KHF in [15]. It is natural to compare the results of [15] to the ones obtained in this section.

(2) Another relevant duality was established by Kakutani [78, 79], the Krein brothers [80], and Yosida [114], who also worked with continuous real-valued functions, but their signature was that of a vector lattice instead of an ℓ-algebra. Gelfand duality has a natural counterpart in this setting. Let $\boldsymbol{bav}$ be the category of bounded archimedean vector lattices and let $\boldsymbol{ubav}$ be its reflective subcategory consisting of uniformly complete objects. Then there is a dual adjunction between $\boldsymbol{bav}$ and KHaus, which restricts to a dual equivalence between $\boldsymbol{ubav}$ and KHaus. This duality is known as Yosida duality (or Kakutani-Krein-Yosida duality). In our axiomatization of $\boldsymbol{mba\ell}$ (see Definition 5.16), the only axiom involving multiplication is

(M5). In the serial case, (M5) simplifies to (M5′) of Remark 5.18, which only involves scalar multiplication. In the non-serial case, (M5) can be replaced by the following two axioms

- $\Box(\lambda a) = \lambda\Box a + (1 - \lambda)\Box 0$ provided $\lambda \geq 0$,
- $\Box 0 \wedge (1 - \Box a)^+ = 0$,

which again only involve vector lattice operations. This yields the category ***mbav*** of modal bounded archimedean vector lattices and its reflective subcategory ***mubav*** consisting of uniformly complete objects. The results of Section 5.4 then generalize to the setting of ***mbav*** and ***mubav***, and provide a generalization of Yosida duality.

(3) Our definition of a modal operator on a bounded archimedean ℓ-algebra can be further adjusted to the settings of ℓ-rings, ℓ-groups, and MV-algebras. In this regard, it would be interesting to develop logical systems corresponding to these algebras.

(4) The theory of canonical extensions originates from the work of Jónsson and Tarski [77] on boolean algebras with operators. Canonical extensions of bounded archimedean ℓ-algebras were introduced in [27]. In [23] we provide a point-free construction of canonical extensions in $\boldsymbol{ba\ell}$. This we do by first adapting the choice-free construction of canonical extensions of boolean algebras of [31] to a point-free construction that we then generalize to $\boldsymbol{ba\ell}$.

(5) It is well known that the category of Kripke frames (see Section 5.2) is isomorphic to $\mathsf{Coalg}(\mathcal{P})$ where $\mathcal{P}$ is the covariant powerset functor on the category of sets. In [19] we define an endofunctor $\mathcal{H}$ on the category of complete and atomic boolean algebras such that $\mathsf{Coalg}(\mathcal{P})$ is dually equivalent to $\mathsf{Alg}(\mathcal{H})$. As a consequence, we obtain that the category KF of Kripke frames is dually equivalent to the category ***cama*** of complete and atomic modal algebras with completely multiplicative $\Box$. This yields an alternate proof of Thomason

duality between KF and $\boldsymbol{cama}$ that is analogous to the alternate proof of Jónsson-Tarski duality of [84]. The category Sets of sets is dually equivalent to the subcategory $\boldsymbol{balg}$ of $\boldsymbol{ba\ell}$ given by the *basic algebras* (see [28, Sec. 3]). In a future work we will extend this duality to a duality between KF and the subcategory $\boldsymbol{mbalg}$ of $\boldsymbol{mba\ell}$ given by the basic algebras with a completely multiplicative modal operator. We will also show that such a duality can be obtained via algebraic/coalgebraic methods by an approach similar to the one employed in this section. Thus, we will obtain a diagram connecting the various approaches to Thomason duality analogous to the one at the end of Section 6.7.

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